Math 55: Midterm #2, 5 April 2001

Problem 1: Eight pecans are given to seven squirrels, with each pecan being given to one squirrel chosen independently at random.
(a) Compute the probability that each squirrel receives at least one pecan. (6)
(b) Compute the expected number of squirrels who receive pecans. (6)
(c) Compute the expected number of pecans received by the first squirrel. (7)
(d) Compute the variance of the number of pecans received by the first squirrel. (6)

Solution: **(a)** This is the number of onto functions from an 8-element set to a 7-element set, divided by the total number of functions. It can be directly evaluated by choosing a squirrel to receive two pecans, choosing two pecans for that squirrel to get, and permuting the other 6 pecans/squirrels:

$$p = \frac{7\binom{8}{2}6!}{7^8}.$$

A common wrong answer is

$$\binom{7}{1} / \binom{8+7-1}{8},$$

which is wrong because the final configurations don't have equal probabilities.
(b) Let f_i be the indicator of the event F_i that the ith squirrel gets a pecan, so we are asked for the expectation of $f = f_1 + \cdots + f_7$. By symmetry, this is $7E(f_1) = 7p(F_1) = 7(1 - p(\bar{F}_1))$. Since there are 6^8 ways for the first squirrel not to get a pecan,

$$E(f) = 7E(f_1) = 7(1 - (6/7)^8).$$

(c) This is the expected number of successes in $n = 8$ Bernoulli trials with success probability $p = 1/7$, which is $np = 8/7$.
(d) This is the variance of the number of successes in $n = 8$ Bernoulli trials with success probability $p = 1/7$, which is $npq = 48/49$.

Problem 2: A discrete math midterm consists of 4 independent true–false questions on Martian literature. On each question, any non-Martian student has a 2/3 chance of guessing the correct answer. One student (in the class of 82 students) is Martian, and therefore gets a perfect score.
(a) What is the probability that a randomly chosen non-Martian student gets a perfect score? (5)
(b) What is the probability that a randomly chosen student gets a perfect score? (Hint: think about conditional probabilities.) (10)
(c) Given that a student named Zřthjkp got a perfect score on the midterm, compute the probability that Zřthjkp is a Martian. (10)

Solution: **(a)** A non-Martian student gets a perfect score with probability $(2/3)^4$.
(b) Let P be the event of a perfect score, M the event that the student is Martian, N the event that the student is not Martian. Since N and M partition the sample space into a disjoint union,

$$p(P) = p(P \cap N) + p(P \cap M) = p(P|N)p(N) + p(P|M)p(M)$$

by definition of conditional probability. Since the conditionals are given, we have

$$p(P) = (2/3)^4 \cdot (81/82) + 1 \cdot (1/82) = 17/82.$$

(c) By (b) and the definition of conditional probability,

$$p(M|P) = \frac{p(M \cap P)}{p(P)} = \frac{p(P|M)p(M)}{p(P)} = \frac{1 \cdot 1/82}{17/82} = \frac{1}{17}.$$

Problem 3: **(a)** Write down the inclusion-exclusion formula for the cardinality of the union $P \cup Q \cup R$ of three sets P, Q and R. (6)

(b) Let p, q and r be three distinct positive prime integers. How many integers n with $1 \le n \le pqr$ are relatively prime to pqr? (12)
(c) Let $p = 2$, $q = 3$ and $r = 5$. Write down all integers n with $1 \le n \le pqr$ which are relatively prime to pqr, and verify that your formula in (b) gives the correct number. (7)

Solution: **(a)**

$$|P \cup Q \cup R| = |P| + |Q| + |R| - |P \cap Q| - |P \cap R| - |Q \cap R| + |P \cap Q \cap R|.$$

(b) Let P be the set of integers between 1 and pqr inclusive which are multiples of p, and define Q and R similarly. Then the answer is $pqr - |P \cup Q \cup R|$, which by inclusion-exclusion is

$$pqr - \lfloor pqr/p \rfloor - \lfloor pqr/q \rfloor - \lfloor pqr/r \rfloor + \lfloor pqr/pq \rfloor + \lfloor pqr/pr \rfloor + \lfloor pqr/qr \rfloor - \lfloor pqr/pqr \rfloor$$

since p, q and r are prime. This works out to $pqr - qr - pr - pq + r + q + p - 1$.
(c) Our formula gives $pqr - qr - pr - pq + r + q + p - 1 = 8 = |\{1, 7, 11, 13, 17, 19, 23, 29\}|$.

Problem 4: For any integer $k \ge 1$, let S_k be the set of all k–letter strings of Xs, Ys and Zs that do not have two consecutive Xs. For example, XYZ is an element of S_3. Let $s_k = |S_k|$.
(a) Evaluate s_1, s_2 and s_3 from the definition of S_k. (5)
(b) Relate S_k to S_{k-1} and S_{k-2} to find a linear homogeneous recurrence relation for s_k. (5)
(c) Given that $s_0 = 1$, verify that your recurrence relation gives the correct result for s_2 and s_3. (5)
(d) Find a closed form (like e^x/x or $1/(1-x)$) for the generating function

$$G(x) = \sum_{k=0}^{\infty} s_k x^k.$$

(10)

Solution: **(a)** By inspection, $s_1 = |\{X, Y, Z\}| = 3$ and

$$s_2 = |\{XY, XZ, YX, YY, YZ, ZX, ZY, ZZ\}| = 8.$$

For $k = 3$, we must exclude 6 strings like XXa and aXX from the 27 possible three-letter strings. However, this excludes XXX twice, so $s_k = 27 - 6 + 1 = 22$ by inclusion-exclusion.
(b) Strings in S_k look like one of four possibilities: $XYaaaaa$, $XZaaaaa$, $Yaaaaaa$ and $Zaaaaaa$, where there are no double Xs in the $aaaa$ string. Hence

$$s_k = 2s_{k-1} + 2s_{k-2}.$$

(c) $s_2 = 2s_1 + 2s_0 = 2 \cdot 3 + 2 \cdot 1 = 8$ and $s_3 = 2 \cdot 8 + 2 \cdot 3 = 22$.
(d) From the recurrence relation, we have

$$\sum_{k=2}^{\infty} s_k x^k = 2\sum_{k=2}^{\infty} s_{k-1} x^k + 2\sum_{k=2}^{\infty} s_{k-2} x^k = 2x\sum_{k=1}^{\infty} s_k x^k + 2x^2 \sum_{k=0}^{\infty} s_k x^k.$$

Adding back the missing terms to complete $G(x)$ on both sides gives

$$G(x) - s_0 - s_1 x = 2x(G(x) - s_0) + 2x^2 G(x),$$

so solving for $G(x)$ gives the closed form

$$G(x) = \frac{1+x}{1-2x-2x^2}.$$