Math 55: Final Exam, 11 May 2001, 12:30-3:30 pm

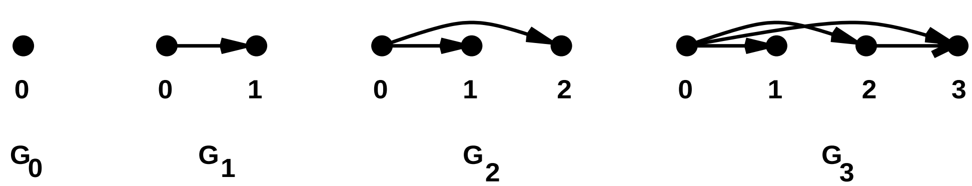
Write your name, your student ID number, your section time and number, a grading grid, and your GSI's name on the cover of your blue book. Books, notes, calculators, scratch paper and/or collaboration are not allowed. Remain in your seat and hand in your exam book *to your GSI* at 3:30 pm. You may leave your answers in terms of binomial coefficients or factorials where convenient. If you finish early, check over your work — do not leave early!

1	
2	
3	
4	
5	
Total	

Problem 1: Choose a positive integer solution (x_1, x_2, x_3) of $x_1 + x_2 + x_3 = 23$ at random, where each solution has equal probability.
(a) What is the probability of selecting $(3, 10, 10)$?
(b) What is the probability that at least one of the x_i's is exactly equal to 10?
(c) What is the probability that $x_1 = 3$, given that $x_2 = 10$?

Problem 2:
(a) Compute 2310^{55} mod 17.
(b) Find the smallest positive inverse of 10 mod 17.
(c) Compute 2310^{55} mod 170.
(d) Find the smallest prime number which divides $2310^{55} - 1$.

Problem 3: Define a directed graph $G_n = (V_n, E_n)$ for nonnegative integers n as follows. G_0 has one vertex named 0 and no edges. Given $G_n = (V_n, E_n)$, we form V_{n+1} by adding one more vertex $n + 1$. The edge set E_{n+1} is E_n plus edges $(j, n + 1)$ for all *even* $j \in V_n$. The first few look like this:



(a) Find the smallest partial order $\preceq$ on V_n whose digraph contains G_n and describe it concretely: $i \preceq j$ iff ...
(b) Draw the Hasse diagram for $\preceq$ when $n = 4$.
(c) Define maximal elements of a poset and determine all maximal elements of the poset $(V_n, \preceq)$.
(d) Derive a first-order recurrence for the number e_n of edges in G_n.

(e) Derive a second-order recurrence for the number e_n of edges in G_n.
(f) Solve your second-order recurrence and check the results for $n \leq 3$.
(g) Use the recurrence relation to find an explicit formula with no $\sum$ or $\ldots$ for the generating function

$$G(x) = \sum_{n=0}^{\infty} e_n x^n.$$

Problem 4: **(a)** Suppose g_n is a given arbitrary sequence of integers and W_n, E_n are two sequences satisfying $W_0 = E_0$ and

$$W_{n+1} \leq 2W_n + g_n, \qquad E_{n+1} = 2E_n + g_n \qquad \text{for } n \geq 0.$$

Prove that $W_n \leq E_n$ for $n \geq 0$.
(b) The worst-case complexity of bilge-sorting n records satisfies

$$W_n \leq 2W_{\lceil n/2 \rceil} + 2W_{\lfloor n/2 \rfloor} + n^2.$$

Prove that $W_n = O(n^2 \log n)$ for n a power of 2 and find explicit constants (from the definition of big-O).

Problem 5: Define a relation R on simple graphs $G = (V, E)$ by the requirement that $G_1 R G_2$ iff $|V_1| = |V_2|$, $|E_1| = |E_2|$, and there is a bijection $f : V_1 \to V_2$ such that for every $v \in V_1$, $\deg(f(v)) = \deg(v)$.
(a) State the definition of an equivalence relation and prove that R is an equivalence relation.
(b) Define isomorphism of simple graphs and prove that if G_1 is isomorphic to G_2 then $G_1 R G_2$.
(c) Find two nonisomorphic simple graphs G_1 and G_2 such that $G_1 R G_2$.
(d) Draw a representative of each equivalence class E of R, for simple graphs with $n = 4$ vertices and $e = 0$ to 4 edges.
(e) Fix $n = 4$ vertices and consider the experiment of selecting an equivalence class E at random with equal probabilities. Let f be the random variable which is 1 if every graph in E contains an Euler circuit and 0 otherwise. State general formulas for the expectation and variance of f. Compute $E(f)$ and $V(f)$.
(f) With the setup of (e), let g be the random variable which is 1 when every graph in E contains a Hamilton circuit and 0 otherwise. State the definition of independence for random variables: are f and g independent?