Math 55: Midterm #2 Solutions, 29 October 1998

Solution 1:

(a) The inclusion-exclusion formula reads

$$|P \cup Q \cup R| = |P| + |Q| + |R| - |P \cap Q| - |P \cap R| - |Q \cap R| + |P \cap Q \cap R|.$$

(b) Define

$$P = \{1 \le n \le pqr \mid p|n \}$$

and similarly for Q and R. Then $|P| = \lfloor pqr/p \rfloor = qr$, while $|Q| = pr$ and $|R| = pq$. Similarly $|P \cap Q| = \lfloor pqr/pq \rfloor = r$ since p, q and r are all distinct primes, and $|P \cap R| = q$, $|Q \cap R| = p$. Finally $|P \cap Q \cap R| = 1$ so we can apply inclusion-exclusion:

$$\begin{aligned} |\{1 \le n \le pqr \mid \gcd(n, pqr) = 1\}| &= pqr - |\{1 \le n \le pqr \mid p|n \vee q|n \vee r|n \}| \\ &= pqr - |P \cup Q \cup R| \\ &= pqr - qr - pr - pq + r + q + p - 1. \end{aligned}$$

(c) Putting $p = 2$, $q = 3$, $r = 5$ gives 8 numbers 1, 7, 11, 13, 17, 19, 23, 29 between 1 and $pqr = 30$ inclusive and relatively prime to 30. Our formula gives

$$pqr - qr - pr - pq + r + q + p - 1 = 30 - 15 - 10 - 6 + 5 + 3 + 2 - 1 = 8.$$

Solution 2:

(a) Since F is all of S except for 11, 33, 55, 77, 99, it has cardinality 45 and $p(F) = 45/50 = 9/10 = 0.9$.

(b) Since expectation is linear, $E(f) = E(d_2) + E(d_1)$. Since the digits are equally probable, $E(d_2) = (0 + 1 + 2 + \cdots + 9)/10 = 4.5$ and $E(d_1) = (1 + 3 + 5 + 7 + 9)/5 = 5$. Summing up, $E(f) = 9.5$.

(c) If $f(n) = 10$ then n is one of the 5 numbers 19, 37, 55, 73 and 91. Only 55 is not in F, so

$$p(G|F) = \frac{p(G \cap F)}{p(F)} = \frac{4/50}{45/50} = \frac{4}{45}.$$

Solution 3:

(a) For $n = 1$, there are three 1-letter strings, and none of them have two consecutive As. Thus $a_1 = 3$. For $n = 2$, there are $3^2 = 9$ 2-letter strings, so excluding AA we find $a_2 = 8$. For $n = 3$, there are $3^3 = 27$ 3-letter strings, of which we must exclude 3 strings of the form AAx and three of the form xAA where x is A, B or C. Howver, this excludes AAA twice, so adding it back we get $a_3 = 27 - 3 - 3 + 1 = 22$.

(b) A string in Y_n must begin with either AB, AC, B or C and continue without consecutive As, so

$$\begin{aligned} a_n &= |\{x_1 x_2 x_3 \ldots x_n\}| \\ &= |\{ABx_3x_4 \ldots x_n\}| + |\{ACx_3x_4 \ldots x_n\}| + |\{Bx_2x_3 \ldots x_n\}| + |\{Cx_2x_3 \ldots x_n\}| \end{aligned}$$

and $a_n = 2(a_{n-1} + a_{n-2})$.

(c) Our relation gives $a_2 = 2(a_1 + a_0) = 2(3 + 1) = 8$ and $a_3 = 2(a_2 + a_1) = 2(8 + 3) = 22$, which checks.

(d) The characteristic equation is $r^2 - 2r - 2 = 0$ and its roots are

$$r = \frac{2 \pm \sqrt{4+8}}{2} = \frac{1 \pm \sqrt{3}}{2}$$

so $r_1 = 1 + \sqrt{3}$ and $r_2 = 1 - \sqrt{3}$. Thus a_n is a linear combination $sr_1^n + tr_2^n$ determined by the initial conditions

$$s + t = 1, \qquad sr_1 + tr_2 = 3.$$

Solving these equations gives

$$s = \frac{1}{2} + \frac{1}{\sqrt{3}}, \qquad t = \frac{1}{2} - \frac{1}{\sqrt{3}},$$

and $a_n = \left(\frac{1}{2} + \frac{1}{\sqrt{3}}\right)\left(1 + \sqrt{3}\right)^n + \left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right)\left(1 - \sqrt{3}\right)^n$. Checking, we get

$$\begin{aligned} a_2 &= \left(\frac{1}{2} + \frac{1}{\sqrt{3}}\right)\left(1 + \sqrt{3}\right)^2 + \left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right)\left(1 - \sqrt{3}\right)^2 \\ &= \left(\frac{1}{2} + \frac{1}{\sqrt{3}}\right)\left(4 + 2\sqrt{3}\right)^2 + \left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right)\left(4 - 2\sqrt{3}\right)^2 = 8. \end{aligned}$$

Solution 4:

(a) This procedure computes the number s_1 of nonnegative integer solutions of

$$x_1 + x_2 + x_3 + x_4 + x_5 = 832,$$

which is the number of ways to choose 832 objects of 5 kinds. By stars and bars, the number is

$$\binom{832 + 5 - 1}{5 - 1} = \binom{836}{4}.$$

(b) Now we are imposing the constraints

$$x_1 \geq 1, \quad x_2 \geq 2, \quad x_3 \geq 3, \quad x_4 \geq 4$$

and $0 \leq x_5 \leq 3$. The first four constraints can be handled by selecting j objects of kind j for $1 \leq j \leq 4$, leaving 822 objects of 5 kinds to select subject to the constraint on x_5. There are two ways to count these possibilities, and they give

$$\binom{825}{3} + \binom{824}{3} + \binom{823}{3} + \binom{822}{3} = \binom{826}{4} - \binom{821}{4} = 462846965.$$