Math 55: Midterm #2, 29 October 1998

Write your name, your student ID number, your section time and number, your GSI's name and a grading grid (see right) on the cover of your blue book. Books, notes, calculators, scratch paper and/or collaboration are not allowed. Remain in your seat and hand in your exam book *to your GSI* promptly at 5:00 pm; solutions will be available after all exams have been collected. If you finish early, check over your work — do not leave early!

1	
2	
3	
4	
Total	

Problem 1 (25 points): Let p, q and r be distinct primes.

(a) Write down the inclusion-exclusion formula for the cardinality of the union $P \cup Q \cup R$ of three sets P, Q and R.

(b) How many integers n with $1 \leq n \leq pqr$ are relatively prime to pqr?

(c) Let $p = 2$, $q = 3$ and $r = 5$. Write down all integers n with $1 \leq n \leq pqr$ and relatively prime to pqr, and check your formula gives the correct number of them.

Problem 2 (25 points): Let the sample space S be the set of two-digit *odd* integers $n = d_2 \cdot 10 + d_1$ with $01 \leq n \leq 99$, chosen at random with equal probabilities.

(a) Find the probability of the event F that an element of S has distinct digits $d_1 \neq d_2$.

(b) Define a random variable f to be the sum of the digits of n: thus $f(n) = d_1 + d_2$. Find the expectation $E(f)$.

(c) Let G be the event that $f(n) = 10$ and F as in (a). Find the conditional probability $p(G|F)$.

Problem 3 (25 points): For any integer $n \geq 1$ let Y_n be the set of all n-letter strings of As, Bs and Cs that do not have two consecutive As. For example, ABCABA is an element of Y_6. Let $a_n = |Y_n|$.

(a) Evaluate a_1, a_2 and a_3 from the definition of Y_n.

(b) Find a linear homogeneous recurrence relation for a_n.

(c) Given that $a_0 = 1$, check that your recurrence relation gives the correct result for a_2 and a_3.

(d) Find an explicit non-recursive formula for a_n valid for all $n \geq 0$.

Problem 4 (25 points): Express the results s_1 and s_2 of the following procedures as binomial coefficients or sums or differences of binomial coefficients.

(a)

procedure atherwood
    $s_1 := 0$
    for $x_1 = 0$ **to** 10000
        for $x_2 = 0$ **to** 10000
            for $x_3 = 0$ **to** 10000
                for $x_4 = 0$ **to** 10000
                    for $x_5 = 0$ **to** 10000
                        if $x_1 + x_2 + x_3 + x_4 + x_5 = 832$ **then** $s_1 := s_1 + 1$
    return s_1

(b)

procedure sarif
    $s_2 := 0$
    for $x_1 = 1$ **to** 10000
        for $x_2 = 2$ **to** 10000
            for $x_3 = 3$ **to** 10000
                for $x_4 = 4$ **to** 10000
                    for $x_5 = 0$ **to** 3
                        if $x_1 + x_2 + x_3 + x_4 + x_5 = 832$ **then** $s_2 := s_2 + 1$
    return s_2