Math 55: Final Exam, 16 December 1998, 12:30-3:30 pm

Multiple choice problems 1–5 are worth 5 points each; list all correct answers. Problems 6–10 are worth 15 points each. Write your name, your student ID number, your section time and number, a ten-problem grading grid (see right), and your GSI's name on the cover of your blue book. Books, notes, calculators, scratch paper and/or collaboration are not allowed. Remain in your seat and hand in your exam book *to your GSI* at 3:30 pm; solutions will be available after all exams have been collected. When you finish, check over your work — do not leave early!

1	
2	
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Total	

Problem 1: The smallest prime number that divides $2250^{666} - 1$ is
(a) 3, **(b)** 5, **(c)** 7, **(d)** 9, **(e)** 11.

Problem 2: The number of nonnegative integer solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 < 10$$

is **(a)** 10^6, **(b)** $\binom{16}{10}$, **(c)** $\binom{15}{10}$, **(d)** $\binom{15}{9}$, **(e)** none of the above.

Problem 3: The number of ways to choose 4 cards from a standard 52-card deck and receive at least one club is

(a) $\binom{13}{1}\binom{39}{3} + \binom{13}{2}\binom{39}{2} + \binom{13}{3}\binom{39}{1} + \binom{13}{4}\binom{39}{0}$,
(b) $\binom{13}{1}\binom{51}{3} + \binom{13}{2}\binom{50}{2} + \binom{13}{3}\binom{49}{1} + \binom{13}{4}\binom{48}{0}$,
(c) $\binom{13}{1}\binom{51}{3}$, **(d)** $\binom{52}{4} - \binom{39}{1}$, **(e)** $\binom{52}{4} - \binom{39}{4}$,

Problem 4: Four balls are placed successively into four cells, with equal probabilities. Given that the first two balls are in different boxes, what is the probability that some box contains exactly three balls?
(a) 1/4, **(b)** 1/8, **(c)** $2/\binom{4}{2}$, **(d)** $\binom{2}{2}/\binom{4}{2}$, **(e)** none of the above.

Problem 5: Arrange the digits $1, 2, 3 \ldots, n$ in random order. The probability that 123 appear in order is
(a) $1/n$, **(b)** $1/n(n-1)$, **(c)** $1/n(n-1)(n-2)$. **(d)** $(n-3)!/\binom{n}{3}$, **(e)** $(n-2)!/\binom{n}{2}$,

Problem 6: **(a)** List all equivalence relations on $\{1, 2, 3\}$.
(b) Let E be the set of partitions of $\{1, 2, 3\}$ into disjoint nonempty subsets. Let the partial order $\preceq$ on E be defined by

$$\forall p \in E \; \forall q \in E \;\; (\, p \preceq q \longleftrightarrow \forall A \in p \; \exists B \in q \;\; (A \subseteq B) \,) \, .$$

Draw the digraph and the Hasse diagram of the relation $\preceq$.
(c) List E in a topologically sorted order.

Problem 7: The cycle C_n is the simple graph on vertices $V = \{1, 2, 3, \ldots, n\}$ with edges $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \ldots, \{n-1, n\}, \{n, 1\}\}$. The *cocycle* $\bar{C}_n$ is its complement, with vertices $\bar{V}$ and edges $\bar{E}$. Justify your answer to each of the following questions.
(a) Determine the cardinality of $\bar{E}$.
(b) For which n are C_n and $\bar{C}_n$ isomorphic?
(c) For which n does $\bar{C}_n$ have an Euler circuit?
(d) For which n does $\bar{C}_n$ have a Hamilton circuit?

Problem 8: Define the divisibility relation R on $\mathbf{Z}_n = \{1, 2, 3, 4, \ldots, n\}$ by $aRb \leftrightarrow a|b$.
(a) Define a partial order and prove or disprove that R is one.
(b) Let M be the matrix of R. Show that the number $d(j)$ of divisors of any integer $j \in \mathbf{Z}_n$ is given by the sum of the entries in column j of M:

$$d(j) = \sum_{i=1}^{n} M_{ij}.$$

(c) Evaluate $d(p)$ for any prime $p \in \mathbf{Z}_n$ and $d(2^k)$ for any $2^k \in \mathbf{Z}_n$.
(d) Consider the experiment of selecting an integer j from $\mathbf{Z}_n$ at random, with equal probabilities. Show that

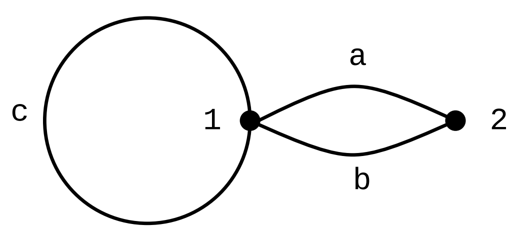
$$E(d) \leq \sum_{k=1}^{n} \frac{1}{k}.$$

Problem 9: Consider the following algorithm:

```
procedure S( a, b : positive integers )
c := 1
while a ≠ b
        if a mod 2 = b mod 2 = 0
                a := a/2
                b := b/2
                c := 2c
        else if a mod 2 = 0 ∧ b mod 2 = 1
                a := a/2
        else if a mod 2 = 1 ∧ b mod 2 = 0
                b := b/2
        else
                d := |a − b|
                b := min(a, b)
                a := d
        end if
end while
return ac
end
```

(a) What does $S(38, 14)$ return?
(b) What function of (a, b) does S compute? Justify your answer.
(c) What is the worst-case complexity of S in terms of a and b? Justify your answer.

Problem 10: Consider the following undirected pseudograph G:



(a) Write down the adjacency matrix A of G.
(b) Find all paths of length 3 from 1 to 2; for example, one is `1c1c1a2`, where the path loops twice at 1 then goes from 1 to 2.
(c) Use the adjacency matrix A to find a recurrence relation for the number b_n of paths from 1 to 2 of any length $n \geq 1$.
(d) Check your recurrence relation predicts the correct value for b_3.
(e) Solve your recurrence relation to obtain an explicit formula for b_n and check your formula gives the correct value for b_1, b_2 and b_3.