Math 55: Final Exam, Dec 18, 1996

Problem 0: Write your name, your student ID number, your section time and number, and your TA's name on the cover of your blue book. Books, notes, calculators, scratch paper and/or collaboration are not allowed. Remain in your seat and hand in your exam book *to your TA* at 3:30 pm; solutions will be available at the exits after all exams have been collected.

Problem 1: Show all steps in the following calculations.
(a) Use the Euclidean algorithm to compute $\gcd(21, 13)$.
(b) Use your work from **(a)** to find integers s and t such that $\gcd(21, 13) = s \cdot 21 + t \cdot 13$.
(c) Use the procedure of the Chinese Remainder Theorem to find an integer x with $0 \le x < 21 \cdot 13 = 273$ such that

$$x \bmod 21 = 5$$

$$x \bmod 13 = 3$$

Problem 2: A computer network consists of $n \ge 10$ computers, each one directly connected to 2 or more of the others.
(a) Prove or give a counterexample: There are at least two computers in the network that are directly connected to the same number of other computers.
(b) Suppose we want to study the network traffic level by sending a packet from computer to computer through the network so that the packet passes through each connection exactly once (in any direction) and returns to its starting point. State a simple condition on the number of connections to each computer which is necessary and sufficient for this test to be possible.

Problem 3: Find a formula for the number of triples (x, y, z) of nonnegative integers satisfying

$$x + y + z = 16$$

(a) and no other restrictions,
(b) subject to

$$x \ge 4 \ \wedge \ y \ge 4 \ \wedge \ z \ge 4,$$

(c) subject to

$$x \le 6 \ \wedge \ y \le 6 \ \wedge \ z \le 6.$$

Problem 4: Let F_n be the nth Fibonacci number defined by $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$ for $n \ge 1$. (They are 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, etc.)
(a) Use induction to prove that $\gcd(F_{n+1}, F_n) = 1$ for $n \ge 1$.
(b) For $n \ge 1$, let $E(n)$ be the number of "$x \bmod y$" calculations required by the Euclidean algorithm to compute $\gcd(F_{n+1}, F_n)$. Use induction to prove that $E(n) = O(n)$.

Problem 5: Let $X = \{a, b, c\}$ and let S be the set of all equivalence relations on S. Consider S as a sample space with uniform probability distribution. Let g (respectively f) be the random variable which assigns to an equivalence relation R the cardinality of its smallest (respectively largest) equivalence class. For example, if R is equality $=$ then $g(R) = f(R) = 1$.

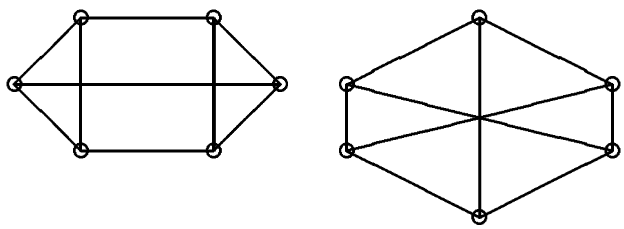
(a) Calculate the cardinality of S.
(b) Calculate $E(f)$ and $V(f)$.
(c) Use Chebyshev's inequality to show that there is a 60% probability that the largest equivalence class of a randomly chosen equivalence relation on X has exactly 2 elements.
(d) Prove or disprove: f and g are independent random variables.

Problem 6: Consider the following pseudocode:

```
function F (n, A = (a1, ..., an), m,  B = (b1, ..., bm), f: A --> B)
do i := 1, m
      s := 0
      do j := 1, n
            if(f(aj) = bi) s := s + 1
      end do
      if(s != 1) return F := False
end do
return F := True
```

(a) What does it compute?
(b) What is its worst-case complexity in terms of `m` and `n` in big-O notation?

Problem 7: Prove or disprove: the following graphs are isomorphic.



Problem 8: Suppose that a randomly chosen child is male with probability 1/2 and female with probability 1/2. Consider two families 1 and 2 with two children each. Let A_1 be the event that family 1 has at least one male child and A_2 be the event that the *oldest* child in family 2 is male. For $i = 1, 2$ let C_i be the event that family number i has two male children.
(a) What is the sample space S? What is the probability $P(x)$ of each point $x \in S$? What is the total expected number of male children in both families?
(b) Calculate the probabilities $P(A_1)$ and $P(A_2)$.
(c) Calculate the conditional probabilities $P(C_1|A_1)$ and $P(C_2|A_2)$. Given that A_1 and A_2 take place, which is more likely: C_1 or C_2?
(d) Define what it means for two events A and B to be independent.
(e) For which i and j are A_i and C_j independent?

Problem 9: Let p and q be propositional variables. Let X be the following set of propositions in the variables p and q:

$$X = \{T, F, p, q, p \wedge q, \neg p \wedge q, p \wedge \neg q, p \vee q, p \oplus q\}.$$

Define a relation R on X by $\alpha R \beta$ iff $\alpha \rightarrow \beta$ is a tautology.
(a) Construct a truth table showing the values of all elements of X (except T and F).
(b) Check that R is a partial order on X.
(c) Construct the Hasse diagram for the poset $P = (X, R)$.
(d) List P in a topologically sorted order.