1 Week 03: 3 February 2009

2 Approximation and interpolation

Interpolation estimates in one and more dimensions are covered exhaustively in Chapter 1 of [EG04].

The simplest estimates control the error in $W = W^{1,p}$ of one-dimensional continuous degree-d piecewise-polynomial Lagrange interpolation of functions $u \in W^{1+d,p}$. The Hölder inequality

$$|\int_a^b u(x)v(x)dx| \leq \|u\|_{L^p}\|v\|_{L^q} \qquad \frac{1}{p} + \frac{1}{q} = 1$$

(a key step in the discovery that L^q is the dual of L^p) implies

$$|u(x) - u(y)| = |\int_x^y u'(t)dt| \leq |x-y|^{1/q}(\int_a^b |u'|^p)^{1/p} = |x-y|^{1/q}\|u'\|_{L^p}.$$

Last week we used the case $p = 1$, $q = \infty$. Most finite element analyses require the case $p = q = 2$.

This estimate implies that any $u \in W^{1,p}$ is bounded if $p \geq 1$ and continuous if $p > 1$. If y is a point where $|u(y)|$ achieves its global minimum, then

$$\int_a^b |u(t)|^p dt \geq |b-a||u(y)|^p$$

or

$$|u(y)| \leq |b-a|^{-1/p}\|u\|_{L^p},$$

so at any point x,

$$|u(x)| \leq |u(x)-u(y)|+|u(y)| \leq |b-a|^{-1/p}\|u\|_{L^p}+|b-a|^{1/q}\|u'\|_{L^p} = |b-a|^{-1/p}\|u\|_{W^{1,q}}.$$

Thus

$$\|u\|_{L^\infty} \leq |b-a|^{-1/p}\|u\|_{W^{1,p}}.$$

First consider the case of linear interpolation $d = 1$. The continuous piecewise-linear interpolant P to equidistant grid values u_j of $u \in W^{1,p}$ is given by

$$P(x) = \theta u_{j+1} + (1-\theta)u_j = \frac{x - x_j}{h}u_{j+1} + \frac{x_{j+1} - x}{h}u_j$$

on the interval $x_j \leq x = \theta x_{j+1} + (1-\theta)x_j < x_{j+1}$. We want to show P is bounded by a constant times u in $W = W^{1,p}$, so the interpolation operator is bounded, and then that the error $u - P$ goes to zero in the norm of W as $h \to 0$. For the first, we have

$$\int_a^b |P(x)|^p dx = h \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} |\theta u_j + (1-\theta)u_{j+1}|^p d\theta \leq h \sum_{j=1}^{N} \|u\|_{L^\infty}^p = \|u\|_{L^\infty}^p,$$

and

$$\int_a^b |P'(x)|^p dx = h^{1-p} \sum_{j=1}^{N} |u_j - u_{j-1}|^p \leq h^{1-p} \sum_{j=1}^{N} (\int_{x_{j-1}}^{x_j} |u'(x)| dx)^p \leq \sum_{j=1}^{N} h^{1-p+p/q} (\int_{x_{j-1}}^{x_j} |u'(x)|^p dx) \leq$$

Summing up and observing that $1 - p + p/q = p(1/p + 1/q - 1) = 0$,

$$\|P\|_{W^{1,p}} \leq \|u\|_{W^{1,p}}$$

so the interpolation operator $u \to P$ is bounded on $W^{1,p}$.

The error estimate requires a higher smoothness assumption

$$u \in W^{2,p} = \{w \in W^{1,p} \,|\, w' \in W^{1,p}\}.$$

Such estimates usually hold if the problem is elliptic and the right-hand side $f \in W^{1,p}$, because then u'' satisfies an elliptic equation with L^p data on the right-hand side.

The estimate is derived by starting with the derivative error

$$e(x) = u'(x) - P'(x),$$

which vanishes at some point z_j on each interval $[x_{j-1}, x_j]$ because $u = P$ at grid points. Thus on each interval,

$$|e(x)|^p = |e(x) - e(z_j)|^p \leq (\int_{z_j}^{x} |e'(t)| dt)^p \leq h^{p/q} \int_{x_{j-1}}^{x_j} |u''(t)|^p dt$$

since $P'' = 0$. Integrating,

$$\|u' - P'\|_{L^p} \leq h \|u''\|_{L^p}.$$

Since the lower-order error $u - P$ vanishes at grid points, the same argument shows that

$$|u(x) - P(x)|^p \leq h^{p/q} \int_{x_{j-1}}^{x_j} |e(x)|^p dx$$

and integration gives

$$\|u - P\|_{L^p} \leq h\|e\|_{L^p} \leq h^2\|u''\|_{L^p}.$$

Adding up the various tems in the W-norm gives

$$\|u - P\|_W \leq Kh\|u\|_{W^{2,p}}.$$

For higher-degree polynomial interpolation, we group the interpolation points into intervals or elements $I_j = [x_{j-1}, x_j]$ where each element carries enough data to uniquely determine a polynomial of degree d, and the data shared between elements guarantees continuity. The error estimate begins with the highest nonzero derivative $P^{[d]}$, which has a zero z_j on each element I_j by Rolle's theorem, and works backward. First observe that the error $e = u^{[d]} - P^{[d]}$ in the highest nonzero derivative satisfies

$$|e(x)|^p = |e(x) - e(z_j)|^p = (\int_{z_j}^{x} |u^{[d+1]}(t)|dt)^p \leq h^{p/q}\int_{z_j}^{x} |u^{[d+1]}(t)|^p dt \leq h^{p/q}\int_{x_{j-1}}^{x_j} |u^{[d+1]}(t)|^p dt.$$

Integrating and summing gives

$$\|u^{[d]} - P^{[d]}\|_{L^p} \leq h\|u^{[d+1]}\|_{L^p},$$

and the rest of the proof works back down through the derivatives as in the linear case. The conclusion is

$$\|u - P\|_W \leq Kh^d\|u\|_{W^{1+d,p}}$$

as expected.

2.1 Hierarchical elements

It is useful to observe that a different choice of basis for the finite-dimensional test function and solution spaces V_h and W_h can improve computational efficiency. With the standard local basis described above, consisting of hat functions for W_h and piecewise constants for V_h, information can travel only from one grid point to the next in one step, requiring $O(N)$ iterative steps to transfer information from one end of the computational domain to the other. Since each step modifies all the data, local iterative methods based on the standard basis are slow – $O(N^2)$ at best. In fact, classical iterations like Jacobi and Gauss-Seidel are lucky to do this well.

Hierarchical bases build the same test functions and solutions from a collection of basis functions on different scales, so information can travel across the entire grid in $O(1)$ or $O(\log N)$ steps. The simplest example uses $N = 2^n$ grid points, and builds solution basis functions hierarchically from the first level $n = 1$, consisting of the s-dimensional space of linear functions $\theta u_N + (1-\theta)u_0$, $\theta = (x-a)/(b-a)$, on the whole interval $[a, b]$ which satisfy the s linearly independent boundary conditions $Au_0 + Bu_N = 0$. The second level adds a hat function of width $(b-a)$ centerd at the interval midpoint $x_{N/2}$, with s additional degrees of freedom $u_{N/2}$, and each additional level $n = 3, \ldots$ adds 2^{n-2} additional hat functions centered at the new grid points equidistant between old grid points. The dimension of W_h is $Ns = 2^n s$. The dual space V_h is built from piecewise constant functions, but now they are chosen so the the functions on each level n are orthogonal to basis functions on coarser levels—and thus linearly independent as well. Thus the first basis function is 1 on the whole interval, while the seond one is 1 on the left half and -1 on the right half. Each subsequent basis function alternates between 1 and -1 on a smaller intervalwhere coarser-level functions are constant. After n levels we have $Ns = 2^n s$ degrees of freedom, so $\dim(V_h) = \dim(W_h)$.

The matrix of L_h in this basis has an interesting block structure which turns out to make it well-conditioned. By contrast, the matrix of L_h in the standard basis has a large condition number $\kappa(L_h) = O(1/h)$ as $h \to 0$.

References

[EG04] A. Ern and J.-L. Guermond. *Theory and Practice of Finite Elements.* Springer-Verlag, New York, 2004.