JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 11, No. 2, 1973

A Proof of the Pontryagin Maximum Principle for Initial-Value Problems

R. J. HARTBERGER[1]

Communicated by G. B. Dantzig

Abstract. Many optimization problems in economic analysis, when cast as optimal control problems, are initial-value problems, not two-point boundary-value problems. While the proof of Pontryagin (Ref. 1) is valid also for initial-value problems, it is desirable to present the potential practitioner with a simple proof specially constructed for initial-value problems. This paper proves the Pontryagin maximum principle for an initial-value problem with bounded controls, using a construction in which all comparison controls remain feasible. The continuity of the Hamiltonian is an immediate corollary. The same construction is also shown to produce the maximum principle for the problem of Bolza.

1. Fundamental Result

We wish to find a vector function $\hat{u}(.)$ which minimizes

$$x^0[u(.)](T) \equiv \int_0^T f^0\{t, x[u(.)](t), u(t)\}\, dt$$

subject to

$$(d/dt)\, x[u(.)](t) = f\{t, x[u(.)](t), u(t)\}, \qquad x[u(.)](0) = A_L, \tag{1}$$

where T is a given fixed end time, and A_L is a given vector constant. We shall write x_i, $i = 1,..., n$, as the ith component of the vector x and u_i, $i = 1,..., m$, as the ith component of the vector u. We assume that f^0: $R^{1+n+m} \to R$ and f: $R^{1+n+m} \to R^n$ are continuously differentiable in each of their arguments.

[1] Applied Mathematician, Esso Mathematics and Systems, Florham Park, New Jersey.

The control functions $u(.)$ will have ranges confined to the hyperrectangle

$$K \equiv [u_{1\,\min}, u_{1\,\max}] \times [u_{2\,\min}, u_{2\,\max}] \times \cdots \times [u_{m\,\min}, u_{m\,\max}].$$

We confine our attention to the set Ω of control functions $u(.)$ such that (i) $u(.)$ has a finite number (possibly zero) of discontinuities on the time interval $[0, T]$ and (ii) the range of the function $u(.)$ is contained in the hyperrectangle K.

Each of the functions $u(.)$ in Ω can easily be synthesized electronically or computationally. On the other hand, Ω is not a complete set of functions, for instance in the L_2-norm, so that establishing the existence of optimal control functions within this set is in general impossible. It is hoped that the simplicity of the proof presented below, assuming the existence of an optimal control function in Ω, will motivate further research to find a suitable completion of Ω, in some appropriate norm.

The mathematical prerequisites for the proof to follow are (a) the differentiation of integrals with respect to parameters in the integrand and in the limits of integration (Ref. 2) and (b) the solution of nonhomogeneous linear systems of differential equations via the fundamental, or transition, matrix (Ref. 3).

It is not necessary to predefine the adjoint, or multiplier, functions nor the Hamiltonian, since these functions emerge naturally in the course of the proof.

Theorem 1.1. (*Pontryagin Maximum Principle for an Initial-Value Problem*). Assume that $\hat{u}(.) \in \Omega$ minimizes $x^0[.](T)$ over the class of control functions. Let $\hat{x}(.)$ be the continuous solution of (1) generated by $\hat{u}(.)$. Then, at each point $\tau \in [0, T]$, there exist $n + 1$ multipliers 1, $\lambda(\tau)$ such that, for each $U \in K$,

$$\begin{aligned} 1 \cdot f^0[\tau, \hat{x}(\tau), U] &+ \lambda(\tau) \cdot f[\tau, \hat{x}(\tau), U] \\ &\geqslant 1 \cdot f^0[\tau, \hat{x}(\tau), \hat{u}(\tau-)] + \lambda(\tau) \cdot f[\tau, \hat{x}(\tau), \hat{u}(\tau-)]. \end{aligned}$$

Proof. For arbitrary (fixed) $U \in K$ and $\alpha > 0$ sufficiently small to insure that $\tau - \alpha > 0$ and that $\hat{u}(.)$ is continuous on $(\tau - \alpha, \tau)$, we define the comparison control

$$u(t, \alpha) \equiv \begin{cases} \hat{u}(t), & t \in [0, \tau - \alpha], \\ U, & t \in (\tau - \alpha, \tau], \\ \hat{u}(t), & t \in (\tau, T]. \end{cases} \tag{2}$$

Notice that such an α can be chosen, since $\hat{u}(.)$ has been assumed to have a finite number of discontinuities, and that each of the comparison controls $u(., \alpha)$ is admissible, that is, $u(., \alpha) \in \Omega$.

We also define $x(t, \alpha)$ as the continuous solution of (1) generated by $u(t, \alpha)$, that is,

$$(d/dt)\, x(t, \alpha) = f[t, x(t, \alpha), u(t, \alpha)], \qquad x(t, \alpha) = A_L. \tag{3}$$

Notice that, for each $t \in [0, T]$, $u(t, 0) = \hat{u}(t)$, $x(t, 0) = \hat{x}(t)$ and that in general $x(t, \alpha) = \hat{x}(t)$ only for $t \in [0, \tau - \alpha]$. Since $\hat{u}(.)$ minimizes $x^0[.](T)$ on Ω, we have

$$x^0[u(. , \alpha)]\,(T) \geqslant x^0[\hat{u}(.)]\,(T), \tag{4}$$

so that the derivative from the right at $\alpha = 0$ is nonnegative, that is,

$$\delta x^0(T) \equiv (\partial/\partial\alpha)_{\alpha=0+}\, \{x^0[u(., \alpha)](T)\} \geqslant 0. \tag{5}$$

The existence of $\delta x^0(T)$ is seen by observing that each $x(., \alpha)$ is differentiable from the right at $\alpha = 0$, and that each of f^0 and f has been assumed continuously differentiable in its arguments. Now,

$$x_0[u(\,\cdot, \alpha)](T) = \int_0^{\tau-\alpha} f^0[t, \hat{x}(t), \hat{u}(t)]\, dt + \int_{\tau-\alpha}^{\tau} f^0[t, x(t, \alpha), U]\, dt + \int_{\tau}^{T} f^0[t, x(t, \alpha), \hat{u}(t)]\, dt. \tag{6}$$

Using the notations

$$\nabla_1 f^0 \equiv (\partial f^0/\partial x_i), \qquad i = 1,..., n,$$

and

$$\delta x(t) \equiv (\partial/\partial\alpha)_{\alpha=0+}\, \{x[u(., \alpha)](t)\},$$

(5) and (6) together imply that

$$-f^0[\tau, \hat{x}(\tau), \hat{u}(\tau-)] + f^0[\tau, \hat{x}(\tau), U] + \int_{\tau}^{T} \nabla_1 f^0[t, \hat{x}(t), \hat{u}(t)] \cdot \delta x(t)\, dt \geqslant 0. \tag{7}$$

The notations $\nabla_1 f^0$ used here and $\nabla_1 f$ used below are justified, since f^0 and f have been assumed differentiable in each of their arguments. But, for $t \in (\tau, T)$,

$$x(t, \alpha) = A_L + \int_0^{\tau-\alpha} f[s, \hat{x}(s), \hat{u}(s)]\, ds + \int_{\tau-\alpha}^{\tau} f[s, x(s, \alpha), U]\, ds + \int_{\tau}^{t} f[s, x(s, \alpha), \hat{u}(s)]\, ds. \tag{8}$$

Hence, using the notation

$$\nabla_1 f = (\partial f_i/\partial x_j), \qquad i, j = 1,..., n,$$

where f_i is the ith component of the vector f, we can write, for $t \in (\tau, T]$,

$$\delta x(t) = -f[\tau, \hat{x}(\tau), \hat{u}(\tau-)] + f[\tau, \hat{x}(\tau), U] + \int_\tau^t \nabla_1 f[s, \hat{x}(s), \hat{u}(s)] \cdot \delta x(s)\, ds. \quad (9)$$

Now, we can rewrite (9) in the differential form

$$(d/dt)\, \delta x(t) = \nabla_1 f \cdot \delta x(t), \qquad \delta x(\tau) = f[\tau, \hat{x}(\tau), U] - f[\tau, \hat{x}(\tau), \hat{u}(\tau-)]. \quad (10)$$

But there exists (Ref. 3) a fundamental solution matrix $\Phi(t, \tau)$ such that, for $t \in (\tau, T]$,

$$(d/dt)\, \Phi(t, \tau) = \nabla_1 f \cdot \Phi(t, \tau), \qquad \Phi(\tau, \tau) = I, \quad (11)$$

where I is the $n \times n$ identity matrix. Hence, (9) or (10), being a system of linear homogeneous equations, has the solution

$$\delta x(t) = \Phi(t, \tau) \cdot \{f[\tau, \hat{x}(\tau), U] - f[\tau, \hat{x}(\tau), \hat{u}(\tau-)]\}. \quad (12)$$

Using (12) in Ineq. (7), we see that

$$-f^0[\tau, \hat{x}(\tau), \hat{u}(\tau-)] + f^0[\tau, \hat{x}(\tau), U] + \int_\tau^T \nabla_1 f^0(t) \cdot \Phi(t, \tau)\, dt \cdot \{f[\tau, \hat{x}(\tau), U] - f[\tau, \hat{x}(\tau), \hat{u}(\tau-)]\} \geqslant 0. \quad (13)$$

But we can *split* Ineq. (13) to obtain

$$f^0[\tau, \hat{x}(\tau), U] + \int_\tau^T \nabla_1 f^0(t) \cdot \Phi(t, \tau)\, dt \cdot f[\tau, \hat{x}(\tau), U] \geqslant f^0[\tau, \hat{x}(\tau), \hat{u}(\tau-)] + \int_\tau^T \nabla_1 f^0(t) \cdot \Phi(t, \tau)\, dt \cdot f[\tau, \hat{x}(\tau), \hat{u}(\tau-)]. \quad (14)$$

This induces the definition

$$\lambda(\tau) \equiv \int_\tau^T \nabla_1 f^0[t, \hat{x}(t), \hat{u}(t)] \cdot \Phi(t, \tau)\, dt, \quad (15)$$

with which the desired inequality is proved.

2. Adjoint Functions

The multiplier functions of Pontryagin are chosen from the set of solutions to the adjoints of the state equations. To see that the set of multipliers $\{1, \lambda(\tau)\}$ are in the same class, recall (Ref. 3) that the fundamental solution matrix also has the following property:

$$(d/d\tau)\, \Phi(t, \tau) = -\Phi(t, \tau) \cdot \nabla_1 f[\tau, \hat{x}(\tau), \hat{u}(\tau)]. \tag{11-1}$$

Then, (11) and (11-1), when used to differentiate (15) with respect to τ, yield

$$(d/d\tau)[1, \lambda(\tau)] = -[1, \lambda(\tau)] \begin{bmatrix} 0 & \nabla_1 f^0 \\ 0 & \nabla_1 f \end{bmatrix}. \tag{16}$$

In particular, in the definition (15), we have chosen the solution vector of the adjoint variables with the terminal condition $[1, \lambda(T)] = [1, 0]$.

3. Continuity of the Hamiltonian

Inequality (14) induces the definition of the Hamiltonian

$$H(\tau, U) \equiv f^0[\tau, \hat{x}(\tau), U] + \lambda(\tau) f[\tau, \hat{x}(\tau), U],$$

with which we can rewrite (14) as

$$H(\tau, U) \geqslant H[\tau, \hat{u}(\tau-)]. \tag{17}$$

In particuar, for $U = \hat{u}(\tau+)$, (17) implies that

$$H[\tau, \hat{u}(\tau+)] \geqslant H[\tau, \hat{u}(\tau-)]. \tag{18}$$

But by a construction of new comparison controls $u(., \alpha)$ on the intervals $[0, \tau]$, $(\tau, \tau + \alpha)$, and $[\tau + \alpha, T]$ instead, we can also derive, similarly to the proof above,

$$H(\tau, U) \geqslant H[\tau, \hat{u}(\tau+)]. \tag{17-1}$$

In particular, for $U = \hat{u}(\tau-)$,

$$H[\tau, \hat{u}(\tau-)] \geqslant H[\tau, \hat{u}(\tau+)]. \tag{18-1}$$

Inequalities (18) and (18-1) imply the continuity of the Hamiltonian, namely,

$$H[\tau, \hat{u}(\tau-)] = H[\tau, \hat{u}(\tau+)], \tag{19}$$

even at points of discontinuity of $\hat{u}(.)$.

4. Extension to the Problem of Bolza

In the problem of Bolza, we add an algebraic end-effect correction to the objective function, defining the new objective functional

$$x^0[u(.)](T) \equiv \Psi\{x[u(.)](T)\} + \int_0^T f^0\{t, x[u(.)](t), u(t)\}\, dt,$$

where Ψ is assumed differentiable in each of its n arguments and the equations of motion are Eqs. (1) above. Using the comparison controls $u(., \alpha)$ defined in (2) above, we see that

$$x^0[u(., \alpha)](T) = \Psi\{x[u(., \alpha)](T)\} + \int_0^T f^0\{t, x[u(., \alpha)](t), u(t, \alpha)\}\, dt; \tag{20}$$

and hence, upon defining $\nabla_x \Psi \equiv \partial\Psi/\partial x_i$, $i = 1,..., n$, we see that

$$\begin{aligned} \delta x^0(T) = \nabla_x \Psi[\hat{x}(T)] \cdot \delta x(T) - f^0[\tau, \hat{x}(\tau), \hat{u}(\tau-)] + f^0[\tau, \hat{x}(\tau), U] \\ + \int_\tau^T \nabla_1 f^0 \cdot \delta x(t)\, dt \geqslant 0. \end{aligned} \tag{21}$$

But from (12), we obtain

$$\delta x(T) = \Phi(T, \tau) \cdot \{f[\tau, \hat{x}(\tau), U] - f[\tau, \hat{x}(\tau), \hat{u}(\tau-)]\}, \tag{22}$$

so that we can rewrite (21), using (12) for $\delta x(t)$, after splitting, as

$$\begin{aligned} f^0[\tau, \hat{x}(\tau), U] + \{\nabla_x \Psi[\hat{x}(T)] \cdot \Phi(T, \tau) + \int_\tau^T \nabla_1 f^0 \cdot \Phi(t, \tau)\, dt\} \cdot f[\tau, \hat{x}(\tau), U] \\ \geqslant f^0[\tau, \hat{x}(\tau), \hat{u}(\tau-)] + \{\nabla_x \Psi[\hat{x}(T)] \cdot \Phi(T, \tau) \\ + \int_\tau^T \nabla_1 f^0 \cdot \Phi(t, \tau)\, dt\} \cdot f[\tau, \hat{x}(\tau), \hat{u}(\tau-)]. \end{aligned} \tag{23}$$

Defining

$$\lambda(\tau) \equiv \nabla_x \Psi[\hat{x}(T)] \cdot \Phi(T, \tau) + \int_\tau^T \nabla_1 f^0 \cdot \Phi(t, \tau)\, dt, \tag{24}$$

and recalling (11-1), we again observe that the adjoint functions $\lambda(.)$ satisfy

$$(d/d\tau)\, \lambda(\tau) = -\nabla_1 f^0[\tau, \hat{x}(\tau), \hat{u}(\tau)] - \lambda(\tau) \cdot \nabla_1 f[\tau, \hat{x}(\tau), \hat{u}(\tau)], \tag{25}$$

but with the new terminal condition

$$\lambda(T) = \nabla_x \Psi[\hat{x}(T)]. \tag{26}$$

Again defining

$$H(\tau, U) \equiv f^0[\tau, \hat{x}(\tau), U] + \lambda(\tau) \cdot f[\tau, \hat{x}(\tau), U], \tag{27}$$

we observe that

$$H(\tau, U) \geqslant H[\tau, \hat{u}(\tau-)] \tag{28}$$

for each $U \in K$, and that the Hamiltonian

$$H[\tau, \hat{u}(\tau)] \tag{29}$$

is a continuous function along the optimal control $\hat{u}(.)$.

5. Conclusions

The results above are not as general as those of Pontryagin, but the proofs are considerably more straightforward. Notice that the perturbations of the optimal control are confined to arbitrarily small intervals, while the disturbances in optimal path are uncorrected throughout the remainder of the interval after the initial point of disturbance. It is hoped that these proofs will make the field of optimal control more accessible to mathematical economists and other potential practitioners whose interests lie in initial-value problems.

References

1. Pontryagin, L. S., Boltyanskii, V. G., Gamkrelidze, R. V., and Mishchenko, E. F., *The Mathematical Theory of Optimal Processes*, John Wiley and Sons (Interscience Publishers), New York, 1962.
2. Courant, R., *Differential and Integral Calculus*, Vol. 2, John Wiley and Sons (Interscience Publishers), New York, 1969.
3. Athans, M., and Falb, P. L., *Optimal Control*, McGraw-Hill Book Company, New York, 1966.