1. Let A be an $m \times n$ matrix.

(a) Show that the range of A is perpendicular to the nullspace of A^T.

(b) Use (a) to show that if

$$c = A^T e \quad \text{where } e = [1 1 1 \ldots 1]^T,$$

Then any feasible solution x of

$$Ax = b, \; x \geq 0, \; c^T x = \min$$

is optimal.

(c) Use duality to show (b) without using (a).

2. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ 8 \end{bmatrix}.$$

(a) If $x = \begin{bmatrix} 1 \\ 0 \\ 2/3 \end{bmatrix}$ is a basic optimal solution of

$$Ax = b, \quad x \geq 0, \quad c^T x = \min,$$

then what can c be? Find $d \in \mathbb{R}^3$ such that $d^T c \leq 0$ whenever x is optimal.

(b) Interpret (a) to express a statement about the relative costs of x_1, x_2 and x_3 implied by the optimality of x.

(c) Under what conditions on A can x be optimal for <u>every</u> $c \geq 0$? Are these conditions satisfied above?

3. Minimize $x_1 + x_2 + 6x_3$ subject to

$$x_1 + x_3 = 4, \qquad x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0:$$
$$x_2 + x_3 = 5,$$

(a) Write down the simplex tableau and carry out a Phase I calculation to find an initial basic feasible solution x.

(b) Write down the extended simplex tableau and carry out Phase II to find an optimal solution x.

(c) Maximize $4y_1 + 5y_2$ subject to

$$y_1 \leq 1,\ y_2 \leq 1,\ y_1 + y_2 \leq 6.$$

4. Let A be an $m \times n$ matrix with $\operatorname{rank}(A) = m \leq n$. For any $y \in \mathbb{R}^m$, show that there is a decomposition

$$y = b + b^*$$

where $b = Ax$ for some $x \geq 0$ and

$$A^T b^* \leq 0.$$