1. Let $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$.
Find $x \in \mathbb{R}^n$ which minimizes
$a^T x$ subject to $b^T x = 1$
and $x \geq 0$. Prove your x
is a minimizer.

2. Use the Farkas alternative
to prove that <u>either</u>

$Ax \leq b$ has a solution $x \geq 0$

<u>or</u>

$A^T y \geq 0$, $y^T b < 0$ has a solution $y \geq 0$
but not both.

3. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad c = \begin{bmatrix} c_1 \\ 1 \\ -1 \\ c_4 \end{bmatrix}$$

$$x = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$$

Determine all values (if any) of b_1, b_2, c_1 and c_4 which make x optimal for the linear program

$$\min c^T x, \quad Ax = b, \quad x \geq 0.$$

4. Use the simplex tableau to do a Phase II calculation for the problem of minimizing

$$x_1 - x_2 - x_3$$

subject to $x \geq 0$ and

$$\begin{bmatrix} 2 & -3 & 1 \\ 5 & -9 & 4 \end{bmatrix} x = \begin{bmatrix} 2 \\ 7 \end{bmatrix}.$$

Start with

$$x = \frac{1}{3}\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}.$$

Explain the results of Phase II.

5. Find the value of the two-person zero-sum game with payoff matrix $\begin{bmatrix} 4 & 0 \\ 6 & 3 \end{bmatrix}$.

6. Suppose X and Y are closed and bounded subsets of $\mathbb{R}^n$ and $f: X \times Y \to \mathbb{R}$ is continuous.

(a) Show that

$$\max_{x \in X} \min_{y \in Y} f(x,y) \leq \min_{y \in Y} \max_{x \in X} f(x,y).$$

(b) Suppose a saddle point (x_0, y_0)

$$f(x, y_0) \leq f(x_0, y_0) \leq f(x_0, y),$$

for all $x \in X$ and $y \in Y$. Show that equality holds in (a).

7. Let A be an $n \times n$ symmetric positive definite matrix and B an $m \times n$ matrix with full row rank. Let $r \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. Show that

$$\min_x \frac{1}{2} x^T A x + r^T x$$

$$\text{subject to } \quad Bx = b$$

has a unique solution x^*. Compute x^* from the KKT conditions.

8. Let E_1 and E_2 be disjoint ellipsoids

$$E_j = \{x \in \mathbb{R}^n \mid (x - c_j)^T A_j (\cancel{x} - c_j) \leq 1\}$$

where A_j are symmetric positive definite matrices and $c_j \in \mathbb{R}^n$.

Let x_1 and x_2 be the points in E_1 and E_2 respectively which are closest to each other.

Use the KKT conditions to show that $x_2 - x_1$ is perpendicular to the tangent plane of E_j at each x_j.

9. Given $n \geq 3$ distinct points $x_j \in \mathbb{R}^2$. Consider the problem

$$x = \operatorname{argmin} \max_{1 \leq j \leq n} \|x - x_j\|_2^2.$$

(a) Introduce a slack variable to eliminate the max and derive an equivalent constrained optimization problem.

(b) Write the resulting KKT conditions.

(c) Find the minimizer.

10. Solve the variational problem

$$\min \int_0^1 (x'(t) - x(t))^2 \, dt$$

subj to $\quad x(0) = 0, \; x(1) = 1.$

11. Find the path

$X = (x(t), y(t)) \qquad 0 \le t \le T$

which connects

$$X(0) = (a, 0) \text{ to } X(T) = (b\cos\theta, b\sin\theta)$$

with minimum "radiation exposure"

$$I(x) = \int_0^T \frac{v(t)\,dt}{\sqrt{x(t)^2 + y(t)^2}}$$

where

$$v(t) = \sqrt{x'(t)^2 + y'(t)^2}$$

is the speed.

12. Use the Pontryagin Maximum Principle to solve

$$\max \int_0^T x^2 - x \, dt$$

subj to $\quad x' = \alpha, \; x(0) = 0,$

$\quad 0 \le \alpha \le 1.$

13. Consider the optimal control problem

$$\max \int_0^{\pi/8} 2x^2 - \tfrac{1}{2}\alpha^2 \, dt$$

subj to $\quad x' = \alpha, \quad x(0) = 1, \quad \alpha \in \mathbb{R}.$

(a) Write down the Pontryagin Maximum Principle for this problem.

(b) Deduce the only possible maximal solution.

(c) Is it a solution?

14. Consider a controlled disturbed system

$$x' = Ax + B\alpha + C\delta \qquad x(0) = x_0$$

where α is the usual control but δ is an unwanted disturbance. Find an Euler-Lagrange equation for the maximal disturbance / minimal control of

$$P[\alpha, \delta] = \int_0^T x^T Q x + \alpha^T R \alpha - \delta^T \delta.$$

15. Formulate and solve an interesting problem and explain its relevance to the course material.