10 Calculus of Variations

10.1 Defining the Problem

1. Extremum of an Integral Expression

A very important problem of the differential calculus is to determine for which x values the given function $y(x)$ has extreme values. The calculus of variations discusses the following problem: For which functions has a certain integral, whose integrand depends also on the unknown function and its derivatives, an extremum value? The calculus of variations concerns itself with determining all the functions $y(x)$ for which the integral expression

$$I[y] = \int\limits_a^b F(x, y(x), y'(x), \ldots, y^{(n)}(x))dx \tag{10.1}$$

has an extremum, if the functions $y(x)$ are from a previously given class of functions. Here, we may define some *boundary* and *side conditions* for $y(x)$ and for its derivatives.

2. Integral Expressions of Variational Calculus

There can also be several variables instead of x in (10.1). In this case, the occurring derivatives are partial derivatives and the integral in (10.1) is a multiple integral. In the calculus of variations, the following types of integral expressions are discussed:

$$I[y] = \int\limits_a^b F(x, y(x), y'(x))\, dx, \tag{10.2}$$

$$I[y_1, y_2, \ldots, y_n] = \int\limits_a^b F(x, y_1(x), \ldots, y_n(x), y_1'(x), \ldots, y_n'(x))\, dx, \tag{10.3}$$

$$I[y] = \int\limits_a^b F(x, y(x), y'(x), \ldots, y^{(n)}(x))\, dx, \tag{10.4}$$

$$I[u] = \iint\limits_\Omega F(x, y, u, u_x, u_y)\, dx\, dy. \tag{10.5}$$

Here the unknown function is $u = u(x, y)$, and Ω represents a plane domain of integration.

$$I[u] = \iiint\limits_R F(x, y, z, u, u_x, u_y, u_z)\, dx\, dy\, dz. \tag{10.6}$$

The unknown function is $u = u(x, y, z)$, and R represents a space region of integration. Additionally, boundary values can be given for the solution of a variational problem, at the endpoints of the interval a and b in the one-dimensional case, and at the boundary of the domain of integration Ω in the two-dimensional case. Besides, various further side conditions can be defined, e.g., in integral form or as a differential equation.

A variational problem is called *first-order* or *higher-order* depending whether the integrand F contains only the first derivative y' or higher derivatives $y^{(n)}$ $(n > 1)$ of the function y.

3. Parametric Representation of the Variational Problem

A variational problem can also be posed in *parametric form.* If we consider a curve in parametric form $x = x(t),\ y = y(t)\ \ (\alpha \le t \le \beta)$, then, e.g., the integral expression (10.2) has the form

$$I[x, y] = \int\limits_\alpha^\beta F(x(t), y(t), \dot{x}(t), \dot{y}(t))\, dt. \tag{10.7}$$

10.2 Historical Problems

10.2.1 Isoperimetric Problem

The *general isoperimetric problem* is to determine the plane region with the largest area among the plane regions with a given perimeter. The solution of this problem, a circle with a given perimeter, originates from queen Dido, who was allowed, as legend has it, to take such an area for the foundation of Carthago which she could be surround by one bull's leather. She cut the leather into fine stripes, and formed a circle with them.

A special case of the isoperimetric problem is to find the equation of the curve $y = y(x)$ in a Cartesian coordinate system connecting the points $A(a, 0)$ and $B(b, 0)$ and having the given length l, for which the area determined by the line segment $\overline{AB}$ and the curve is the largest possible **(Fig. 10.1)**. The mathematical formalization is: We have to determine a one-time continuously differentiable function $y(x)$ such that

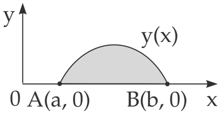

Figure 10.1

$$I[y] = \int_a^b y(x)\,dx = \max \tag{10.8a}$$

holds, where the side condition (10.8b) and the boundary conditions (10.8c) are satisfied:

$$G[y] = \int_a^b \sqrt{1 + y'^2(x)}\,dx = l \qquad (10.8b) \qquad\qquad y(a) = y(b) = 0 \qquad (10.8c)$$

10.2.2 Brachistochrone Problem

The brachistochrone problem was formulated in 1696 by J. Bernoulli, and it is the following: A point mass descends from the point $P_0(x_0, y_0)$ to the origin in the vertical plane x, y only under the influence of gravity. We should determine the curve $y = y(x)$ along which the point reaches the origin in the shortest possible time from P_0 **(Fig. 10.2)**. Considering the formula for the time of fall, T, we get the mathematical description: We have to determine a one-time continuously differentiable function $y = y(x)$, for which

$$T[y] = \int_0^{x_0} \frac{\sqrt{1 + y'^2}}{\sqrt{2g(y_0 - y)}}\,dx = \min, \tag{10.9}$$

(g is the acceleration due to gravity) and the boundary value conditions are

$$y(0) = 0, \qquad y(x_0) = y_0. \tag{10.10}$$

We see that there is a singularity for $x = x_0$ in (10.9).

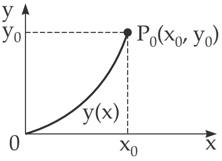

Figure 10.2

10.3 Variational Problems of One Variable

10.3.1 Simple Variational Problems and Extremal Curves

A *simple variational problem* is to determine the extreme value of the integral expression given in the form

$$I[y] = \int_a^b F(x, y(x), y'(x))\,dx, \tag{10.11}$$

where $y(x)$ is a twice continuously differentiable function satisfying the boundary conditions $y(a) = A$ and $y(b) = B$. The values a, b and A, B, and the function F are given.

The integral expression (10.11) is an example of a so-called *functional.* A functional assigns a real number to every function $y(x)$ from a certain class of functions.
If the functional $I[y]$ in (10.11) takes, e.g., its relative maximum for a function $y_0(x)$, then

$$I[y_0] \geq I[y] \tag{10.12}$$

for every twice continuously differentiable function y satisfying the boundary conditions. The curve $y = y_0(x)$ is called an *extremal curve.* Sometimes all the solutions of the Euler differential equation of the variational calculus are called extremal curves.

10.3.2 Euler Differential Equation of the Variational Calculus

We get a necessary condition for the solution of the variational problem in the following way: We construct an *auxiliary curve* or *comparable curve* for the extremal $y_0(x)$ characterized by (10.12)

$$y(x) = y_0(x) + \epsilon\,\eta(x) \tag{10.13}$$

with a twice continuously differentiable function $\eta(x)$ satisfying the special boundary conditions $\eta(a) = \eta(b) = 0$. ϵ is a real parameter. Substituting (10.13) in (10.11) we get a function depending on ϵ instead of the functional $I[y]$

$$I(\epsilon) = \int\limits_a^b F(x, y_0 + \epsilon\,\eta, y_0' + \epsilon\,\eta')\,dx, \tag{10.14}$$

and the functional $I[y]$ has an extreme value for $y_0(x)$ if the function $I(\epsilon)$, as a function of ϵ, has an extreme value for $\epsilon = 0$. Now, we deduce the variational problem to an extreme value problem with the necessary condition

$$\frac{dI}{d\epsilon} = 0 \quad \text{for} \quad \epsilon = 0. \tag{10.15}$$

Supposing that the function F, as a function of three independent variables, is differentiable as many times as needed, by its Taylor expansion we get (see 7.3.3.3, p. 417)

$$I(\epsilon) = \int\limits_a^b \left[F(x, y_0, y_0') + \frac{\partial F}{\partial y}(x, y_0, y_0')\epsilon\,\eta + \frac{\partial F}{\partial y'}(x, y_0, y_0')\epsilon\,\eta' + O(\epsilon^2)\right] dx. \tag{10.16}$$

The necessary condition (10.15) results in the equation

$$\int\limits_a^b \eta \frac{\partial F}{\partial y}\,dx + \int\limits_a^b \eta' \frac{\partial F}{\partial y'}\,dx = 0. \tag{10.17}$$

By partial integration of this equation and considering the boundary conditions for $\eta(x)$, we get

$$\int\limits_a^b \eta \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right) dx = 0. \tag{10.18}$$

From the assumption of continuity and because the integral in (10.18) must disappear for any considerable $\eta(x)$,

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \tag{10.19}$$

must hold. The equation (10.19) gives a *necessary condition for the simple variational problem* and it is called the *Euler differential equation of the calculus of variations.* The differential equation (10.19) can be written in the form

$$\frac{\partial F}{\partial y} - \frac{\partial^2 F}{\partial x \partial y'} - \frac{\partial^2 F}{\partial y \partial y'}y' - \frac{\partial^2 F}{\partial y'^2}y'' = 0. \tag{10.20}$$

It is an ordinary second-order differential equation if $F_{y'y'} \neq 0$ holds.

The Euler differential equation has a simpler form in the following special cases:
Case 1: $F(x, y, y') = F(y')$, i.e., x and y do not appear explicitly. Then instead of (10.19) we get

$$\frac{\partial F}{\partial y} = 0 \qquad (10.21\text{a}) \qquad \text{and} \qquad \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0. \qquad (10.21\text{b})$$

Case 2: $F(x, y, y') = F(y, y')$, i.e., x does not appear explicitly. We consider

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}y'' - y''\frac{\partial F}{\partial y'} - y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = y'\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right) \qquad (10.22\text{a})$$

and because of (10.19), we get

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0, \qquad (10.22\text{b}) \qquad \text{i.e.,} \qquad F - y'\frac{\partial F}{\partial y'} = c \quad (c \text{ const}) \qquad (10.22\text{c})$$

as a necessary condition for the solution of the simple variational problem in the case $F = F(y, y')$.

■ **A:** The functional to determine the shortest curve connecting the points $P_1(a, A)$ and $P_2(b, B)$ in the x, y plane is:

$$I[y] = \int_a^b \sqrt{1 + y'^2}\, dx = \min. \qquad (10.23\text{a})$$

It follows from (10.21b) for $F = F(y') = \sqrt{1 + y'^2}$ that

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{y''}{\left(\sqrt{1 + y'^2}\right)^3} = 0, \qquad (10.23\text{b})$$

so $y'' = 0$, i.e., the shortest curve is the straight line.

■ **B:** We connect the points $P_1(a, A)$ and $P_2(b, B)$ by a curve $y(x)$, and we rotate it around the x-axis. Then the surface area is

$$I[y] = 2\pi \int_a^b y\sqrt{1 + y'^2}\, dx. \qquad (10.24\text{a})$$

For which curve $y(x)$ will the surface area be the smallest? It follows from (10.22c) with $F = F(y, y') = 2\pi y\sqrt{1 + y'^2}$ that $y = \dfrac{c}{2\pi}\sqrt{1 + y'^2}$ or $y'^2 = \left(\dfrac{y}{c_1}\right)^2 - 1$ with $c_1 = \dfrac{c}{2\pi}$. This differential equation is separable (see 9.2.2.3, **1.**, p. 524), and its solution is

$$y = c_1 \cosh\left(\frac{x}{c_1} + c_2\right) \qquad (c_1,\ c_2 \text{ const}), \qquad (10.24\text{b})$$

the equation of the so-called *catenary curve* (see 2.15.1, p. 105). We determine the constants c_1 and c_2 from the boundary values $y(a) = A$ and $y(x) = B$. We have to solve a non-linear equation system (see 19.2.2, p. 896), which cannot be solved for every boundary value.

10.3.3 Variational Problems with Side Conditions

These problems are usually isoperimetric problems (see 10.2.1, p. 553): The simple variational problem (see 10.2.1, p. 553), given by the functional (10.11), is completed by a further side condition in the form

$$\int\limits_a^b G(x, y(x), y'(x))\, dx = l \quad (l \text{ const}) \qquad (10.25)$$

where the constant l and the function G are given. A method to solve this problem is given by Lagrange (extreme values with side conditions in equation form, see 6.2.5.6, p. 403). We consider the expression

$$H(x, y(x), y'(x), \lambda) = F(x, y(x), y'(x)) + \lambda\left(G(x, y(x), y'(x)) - l\right), \qquad (10.26)$$

where λ is a parameter, and we consider the problem

$$\int_a^b H(x, y(x), y'(x), \lambda) = \text{extreme!}, \tag{10.27}$$

i.e., an extreme value problem without side condition. The corresponding Euler differential equation is:

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0. \tag{10.28}$$

The solution $y = y(x, \lambda)$ depends on the parameter λ, which can be determined by substituting $y(x, \lambda)$ into the side condition (10.25).

■ For the isoperimetric problem 10.2.1, p. 553, we get

$$H(x, y(x), y'(x), \lambda) = y + \lambda\sqrt{1 + y'^2}\,. \tag{10.29a}$$

Because the variable x does not appear in H, we get instead of the Euler differential equation (10.28), analogously to (10.22c), the differential equation

$$y + \lambda\sqrt{1 + y'^2} - \frac{\lambda y'^2}{\sqrt{1 + y'^2}} = c_1 \quad \text{or} \quad y'^2 = \frac{\sqrt{\lambda^2 - (c_1 - y)^2}}{c_1 - y} \quad (c_1 \text{ const}), \tag{10.29b}$$

whose solution is the family of circles

$$(x - c_2)^2 + (y - c_1)^2 = \lambda^2 \quad (c_1, c_2, \lambda \text{ const}). \tag{10.29c}$$

The values c_1, c_2 and λ are determined from the conditions $y(a) = 0$, $y(b) = 0$ and from the requirement that the arclength between A and B should be l. We get a non-linear equation for λ, which should be solved by an appropriate iterative method.

10.3.4 Variational Problems with Higher-Order Derivatives

We consider two types of problems.

1. $F = F(x, y, y', y'')$

The variational problem is:

$$I[y] = \int_a^b F(x, y, y', y'')\,dx = \text{extreme!} \tag{10.30a}$$

with the boundary values

$$y(a) = A, \quad y(b) = B, \quad y'(a) = A', \quad y'(b) = B', \tag{10.30b}$$

where the numbers a, b, A, B, A', and B', and the function F are given. Similarly as in 10.3.2, p. 554, we introduce comparable curves $y(x) = y_0(x) + \epsilon\,\eta(x)$ with $\eta(a) = \eta(b) = \eta'(a) = \eta'(b) = 0$, and we get the *Euler differential equation*

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0 \tag{10.31}$$

as a necessary condition for the solution of the variational problem (10.30a). The differential equation (10.31) represents a fourth-order differential equation. Its general solution contains four arbitrary constants which can be determined by the boundary values (10.30b).

■ Consider the problem

$$I[y] = \int_0^1 (y''^2 - \alpha y'^2 - \beta y^2)\,dx = \text{extreme!} \tag{10.32a}$$

with the given constants α and β for $F = F(y, y', y'') = y''^2 - \alpha y'^2 - \beta y^2$. Then:
$F_y = -2\beta y$, $F_{y'} = -2y'$, $F_{y''} = 2y''$, $\frac{d}{dx}(F_{y'}) = -2y''$, $\frac{d^2}{dx^2}(F_{y''}) = -2y^{(4)}$, and the Euler differential equation is

$$y^{(4)} + \alpha y'' - \beta y = 0. \tag{10.32b}$$

This is a fourth-order linear differential equation with constant coefficients (see 9.1.2.3, p. 500).

2. $\boldsymbol{F = F(x, y, y', \ldots, y^{(n)})}$

In this general case, when the functional $I[y]$ of the variational problem depends on the derivatives of the unknown function y up to order n $(n \geq 1)$, the corresponding Euler differential equation is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) - \cdots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) = 0, \tag{10.33}$$

whose solution should satisfy the boundary conditions analogously to (10.30b) up to order $n - 1$.

10.3.5 Variational Problem with Several Unknown Functions

Suppose the functional of the variational problem has the form

$$I[y_1, y_2, \ldots, y_n] = \int_a^b F(x, y_1, y_2, \ldots, y_n, y_1', y_2', \ldots, y_n')\, dx, \tag{10.34}$$

where the unknown functions $y_1(x), y_2(x), \ldots, y_n(x)$ should take given values at $x = a$ and $x = b$. We consider n twice continuously differentiable comparable functions

$$y_i(x) = y_{i0}(x) + \epsilon_i \eta_i(x) \quad (i = 1, 2, \ldots, n), \tag{10.35}$$

where the functions $\eta_i(x)$ should disappear at the endpoints.
(10.34) becomes $I(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ with (10.35), and from the necessary conditions

$$\frac{\partial I}{\partial \epsilon_i} = 0 \quad (i = 1, 2, \ldots, n) \tag{10.36}$$

for the extreme values of a function of several variables, we get the n Euler differential equations

$$\frac{\partial F}{\partial y_1} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_1'}\right) = 0, \quad \frac{\partial F}{\partial y_2} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_2'}\right) = 0, \quad \ldots, \quad \frac{\partial F}{\partial y_n} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_n'}\right) = 0, \tag{10.37}$$

whose solutions $y_1(x), y_2(x), \ldots, y_n(x)$ must satisfy the given boundary conditions.

10.3.6 Variational Problems using Parametric Representation

For some variational problems it is useful to determine the extremal, not in the explicit form $y = y(x)$, but in the parametric form

$$x = x(t), \quad y = y(t) \quad (t_1 \leq t \leq t_2), \tag{10.38}$$

where t_1 and t_2 are the parameter values corresponding to the points (a, A) and (b, B). Then the simple variational problem (see 10.3.1, p. 553) is

$$I[x, y] = \int_{t_1}^{t_2} F(x(t), y(t), \dot{x}(t), \dot{y}(t))dt = \text{extreme!} \tag{10.39a}$$

with the boundary conditions

$$x(t_1) = a, \quad x(t_2) = b, \quad y(t_1) = A, \quad y(t_2) = B. \tag{10.39b}$$

Here $\dot{x}$ and $\dot{y}$ denote the derivatives of x and y with respect to the parameter t, as usual in the parametric representation.

The variational problem (10.39a) makes sense only if the value of the integral is independent of the parametric representation of the extremal curve. To ensure the integral in (10.39a) is independent of the parametric representation of the curve connecting the points (a, A) and (b, B), F must be a *positive homogeneous function* of the degree 1 of homogeneity (2.18.2.5, **4.**, p. 120), i.e.,

$$F(x, y, \mu\dot{x}, \mu\dot{y}) = \mu F(x, y, \dot{x}, \dot{y}) \quad (\mu > 0) \tag{10.40}$$

must hold.
Because the variational problem (10.39a) can be considered as a special case of (10.34), the corresponding Euler differential equations are

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0, \quad \frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right) = 0. \tag{10.41}$$

They are not independent of each other, but they are equivalent to the so-called Weierstrass form of the Euler differential equation:

$$\frac{\partial^2 F}{\partial x \partial \dot{y}} - \frac{\partial^2 F}{\partial \dot{x} \partial y} + M(\dot{x}\ddot{y} - \ddot{x}\dot{y}) = 0 \tag{10.42a}$$

with

with$M = \dfrac{1}{\dot{y}^2}\dfrac{\partial^2 F}{\partial \dot{x}^2} = -\dfrac{1}{\dot{x}\dot{y}}\dfrac{\partial^2 F}{\partial \dot{x}\partial \dot{y}} = \dfrac{1}{\dot{x}^2}\dfrac{\partial^2 F}{\partial \dot{y}^2}.$ (10.42b)

Starting with the calculation of the radius of curvature R of a curve given in parametric representation (see 3.6.1.1, **1.**, p. 225), we calculate the *radius of curvature of the extremal curve* considering (10.42a) with

$$R = \left|\frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \ddot{x}\dot{y}}\right| = \left|\frac{M\,(\dot{x}^2 + \dot{y}^2)^{3/2}}{F_{\dot{x}y} - F_{x\dot{y}}}\right|. \tag{10.42c}$$

■ The isoperimetric problem (10.8a to 10.8c) (see 10.2.1, p. 553) has the form in parametric representation:

$$I[x, y] = \int_{t_1}^{t_2} y(t)\dot{x}(t)dt = \max! \quad (10.43a) \qquad \text{with} \quad G[x, y] = \int_{t_1}^{t_2} \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}\, dt = l. \quad (10.43b)$$

This variational problem with the side condition becomes a variational problem without the side condition according to (10.26) with

$$H = H(x, y, \dot{x}, \dot{y}) = y\dot{x} + \lambda\sqrt{\dot{x}^2 + \dot{y}^2}. \tag{10.43c}$$

We see that H satisfies the condition (10.40), so it is a positive homogeneous function of first degree. Furthermore, we have

$$M = \frac{1}{\dot{y}^2}H_{\dot{x}\dot{x}} = \frac{\lambda}{(\dot{x}^2 + \dot{y}^2)^{3/2}}, \quad H_{\dot{x}y} = 1, \quad H_{x\dot{y}} = 0, \tag{10.43d}$$

so (10.42c) yields that the radius of curvature is $R = |\lambda|$. Since λ is a constant, the extremals are circles.

10.4 Variational Problems with Functions of Several Variables

10.4.1 Simple Variational Problem

One of the simplest problems with a function of several variables is the following variational problem for a double integral:

$$I[u] = \iint\limits_{(G)} F(x, y, u(x, y), u_x, u_y)\, dx\, dy = \text{extreme!} \tag{10.44}$$

Here, the unknown function $u = u(x, y)$ should take given values on the boundary Γ of the domain G. Analogously to 10.3.2, p. 554, we introduce the *comparable function* in the form

$$u(x, y) = u_0(x, y) + \epsilon\eta(x, y), \tag{10.45}$$

where $u_0(x, y)$ is a solution of the variational problem (10.44) and it takes the given boundary values, while $\eta(x, y)$ satisfies the condition

$$\eta(x, y) = 0 \quad \text{on the boundary } \Gamma \tag{10.46}$$

and together with $u_0(x, y)$, they are differentiable as many times as needed.
The quantity ϵ is a parameter. We determine a surface by $u = u(x, y)$ which is close to the solution surface $u_0(x, y)$. $I[u]$ becomes $I(\epsilon)$ with (10.45), i.e., the variational problem (10.44) becomes an extreme value problem which must satisfy the necessary conditions

$$\frac{dI}{d\epsilon} = 0 \qquad \text{for} \quad \epsilon = 0. \tag{10.47}$$

We get from this the *Euler differential equation*

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0 \tag{10.48}$$

as a necessary condition for the solution of the variational problem (10.44).

■ A free membrane, fixed at the perimeter Γ of a domain G of the x, y plane, covers a surface with area

$$I_1 = \iint\limits_{(G)} dx\, dy. \tag{10.49a}$$

If the membrane is deformed by a load so that every point has an elongation $u = u(x, y)$ in the z-direction, then its area is calculated by the formula

$$I_2 = \iint\limits_{(G)} \sqrt{1 + u_x^2 + u_y^2}\, dx\, dy. \tag{10.49b}$$

If we linearize the integrand in (10.49b) using Taylor series (see 6.2.2.3, p. 396), then we get the relation

$$I_2 \approx I_1 + \frac{1}{2}\iint\limits_{(G)} \left(u_x^2 + u_y^2\right)\, dx\, dy. \tag{10.49c}$$

We have

$$U = \sigma(I_2 - I_1) = \frac{\sigma}{2}\iint\limits_{(G)} \left(u_x^2 + u_y^2\right)\, dx\, dy, \tag{10.49d}$$

for the potential energy U of the deformed membrane, where the constant σ denotes the tension of the membrane. We obtain the so-called *Dirichlet variational problem* in this way: We have to determine the function $u = u(x, y)$ so that the functional

$$I[u] = \iint\limits_{(G)} \left(u_x^2 + u_y^2\right)\, dx\, dy \tag{10.49e}$$

should have an extremum, and u vanishes on the boundary Γ of the plane domain G. The corresponding Euler differential equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{10.49f}$$

It is the Laplace differential equation for functions of two variables (see 13.5.1, p. 669).

10.4.2 More General Variational Problems

We should consider two generalizations of the simple variational problem.

1. $\boldsymbol{F = F(x, y, u(x, y), u_x, u_y, u_{xx}, u_{xy}, u_{yy})}$
The functional depends on higher-order partial derivatives of the unknown function $u(x, y)$. If the partial derivatives occur up to second order, then the Euler differential equation is:

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial u_{xx}}\right) + \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial F}{\partial u_{xy}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial u_{yy}}\right) = 0. \qquad (10.50)$$

2. $\boldsymbol{F = F(x_1, x_2, \ldots, x_n, u(x_1, \ldots, x_n), u_{x_1}, \ldots, u_{x_n})}$
In the case of a variational problem with n independent variables $x_1, x_2, \ldots, x_n$, the Euler differential equation is:

$$\frac{\partial F}{\partial u} - \sum_{k=1}^{n} \frac{\partial}{\partial x_k}\left(\frac{\partial F}{\partial u_{x_k}}\right) = 0. \qquad (10.51)$$

10.5 Numerical Solution of Variational Problems

Most often two ways are used to solve variational problems in practice.

1. Solution of the Euler Differential Equation and Fitting the Found Solution to the Boundary Conditions

Usually, exact solution of the Euler differential equation is possible only in the simplest cases, so we have to use a numerical method to solve the boundary value problem for ordinary or for partial differential equations (see 19.5, p. 911 or 20.4.4, p. 992ff).

2. Direct Methods

The direct methods start directly from the variational problem and do not use the Euler differential equation. The most popular and probably the oldest procedure is the *Ritz method.* It belongs to the so-called approximation methods which are also used for approximate solutions of differential equations (see 19.4.2.2, p. 909 and 19.5.2, p. 912), and we demonstrate it with the following example.

■ Solve numerically the isoperimetric problem

$$\int_0^1 y'^2(x)\,dx = \text{extreme!} \quad (10.52\text{a}) \qquad \text{for} \quad \int_0^1 y^2(x)\,dx = 1 \quad \text{and} \quad y(0) = y(1) = 0. \quad (10.52\text{b})$$

The corresponding variational problem without side condition according to 10.3.3, p. 555, is:

$$I[y] = \int_0^1 \left[y'^2(x)\,dx - \lambda y^2(x)\right] = \text{extreme!} \qquad (10.52\text{c})$$

We want to find the best solution of the form

$$y(x) = a_1 x(x-1) + a_2 x^2(x-1). \qquad (10.52\text{d})$$

Both approximation functions $x(x-1)$ and $x^2(x-1)$ are linearly independent, and satisfy the boundary conditions. (10.52c) is reduced with (10.52d) to

$$I(a_1, a_1) = \frac{1}{3}a_1^2 + \frac{2}{15}a_2^2 + \frac{1}{3}a_1 a_2 - \lambda\left(\frac{1}{30}a_1^2 + \frac{1}{105}a_2^2 + \frac{1}{30}a_1 a_2\right), \qquad (10.52\text{e})$$

and the necessary conditions $\frac{\partial I}{\partial a_1} = \frac{\partial I}{\partial a_2} = 0$ result in he homogeneous linear equation system

$$\left(\frac{2}{3} - \frac{\lambda}{15}\right)a_1 + \left(\frac{1}{3} - \frac{\lambda}{30}\right)a_2 = 0, \qquad \left(\frac{1}{3} - \frac{\lambda}{30}\right)a_1 + \left(\frac{4}{15} - \frac{2\lambda}{105}\right)a_2 = 0. \qquad (10.52\text{f})$$

This system has a non-trivial solution only if the determinant of the coefficient matrix is equal to zero. So, we get:

$$\lambda^2 - 52\lambda + 420 = 0, \quad \text{i.e.,} \quad \lambda_1 = 10,\ \lambda_2 = 42. \tag{10.52g}$$

For $\lambda = \lambda_1 = 10$ we get from (10.52f) $a_2 = 0$, a_1 arbitrary, so the normed solution belonging to $\lambda_1 = 10$ is:

$$y = 5.48x(x-1). \tag{10.52h}$$

To make a comparison, consider the Euler differential equation belonging to (10.52f). We get the boundary value problem

$$y'' + \lambda y = 0 \quad \text{with} \quad y(0) = y(1) = 0 \tag{10.52i}$$

with the eigenvalues $\lambda_k = k^2\pi^2$ $(k = 1, 2, \ldots)$ and the solution $y_k = c_k \sin k\pi x$. The normed solution, e.g., for the case $k = 1$, i.e., $\lambda_1 = \pi^2 \approx 9.87$ is

$$y = \sqrt{2}\sin \pi x, \tag{10.52j}$$

which is really very close to the approximate solution (10.52h).

Remark: With today's level of computers and science we have to apply, first of all, the *finite element method* (FEM) for numerical solutions of variational problems.
The basic idea of this method is given in 19.5.3, p. 913, for numerical solutions of differential equations. The correspondence between differential and variational equations will be used there, e.g., by Euler differential equations or bilinear forms according to (19.146a,b).

Also the *gradient method* can be used for the numerical solution of variational problems as an efficient numerical method for non-linear optimization problems (see 18.2.7, p. 871).

10.6 Supplementary Problems

10.6.1 First and Second Variation

In the derivation of the Euler differential equation with a comparable function (see 10.3.2, p. 554), we stopped after the linear term with respect to ϵ of the Taylor expansion of the integrand of

$$I(\epsilon) = \int_a^b F(x,\, y_0 + \epsilon\eta,\, y_0' + \epsilon\eta')\, dx. \tag{10.53}$$

If we consider also quadratic terms, then we get

$$\begin{aligned} I(\epsilon) - I(0) &= \epsilon \int_a^b \left[\frac{\partial F}{\partial y}(x, y_0, y_0')\eta + \frac{\partial F}{\partial y'}(x, y_0, y_0')\eta'\right] dx \\ &+ \frac{\epsilon^2}{2}\int_a^b \left[\frac{\partial^2 F}{\partial y^2}(x, y_0, y_0')\eta^2 + 2\frac{\partial^2 F}{\partial y \partial y'}(x, y_0, y_0')\eta\eta' + \frac{\partial^2 F}{\partial y'^2}(x, y_0, y_0')\eta'^2 + O(\epsilon)\right] dx. \end{aligned} \tag{10.54}$$

If we denote as
1. Variation δI of the functional $I[y]$ the expression

$$\delta I = \int_a^b \left[\frac{\partial F}{\partial y}(x, y_0, y_0')\eta + \frac{\partial F}{\partial y'}(x, y_0, y_0')\eta'\right] dx \ \text{ and as} \tag{10.55}$$

2. Variation $\delta^2 I$ of the functional $I[y]$ the expression

$$\delta^2 I = \int_a^b \left[\frac{\partial^2 F}{\partial y^2}(x, y_0, y_0')\eta^2 + 2\frac{\partial^2 F}{\partial y \partial y'}(x, y_0, y_0')\eta\eta' + \frac{\partial^2 F}{\partial y'^2}(x, y_0, y_0')\eta'^2\right] dx, \tag{10.56}$$

then we can write:

$$I(\epsilon) - I(0) \approx \epsilon\,\delta I + \frac{\epsilon^2}{2}\,\delta^2 I. \tag{10.57}$$

We can formalize the different optimality conditions with these variations for the functional $I[y]$ (see [10.7]).

10.6.2 Application in Physics

Variational calculus has a determining role in physics. We can derive the fundamental equations of Newtonian mechanics from a variational principle and arrive at the Jacobi–Hamilton theory. Variational calculus is also very important in both atomic theory and quantum physics. It is obvious that the extension and generalization of classical mathematical notions is undoubtedly necessary. So, the calculus of variations must be discussed today by modern mathematical disciplines, e.g., functional analysis and optimization. Unfortunately, we can only give a brief account of the classical part of the calculus of variations (see [10.4], [10.5], [10.7]).