Math 128a, Section 3 — Midterm Solutions — October 25, 2001

(47) **Problem 1** Suppose that a numerical integration rule

$$\int_0^1 f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

with points $x_i \in [0,1]$ is exact for polynomials of degree d. Suppose that a function f can be approximated by a polynomial p of degree d to accuracy ϵ on the interval $[0,1]$:

$$\max_{0\le x\le 1} |f(x)-p(x)| \le \epsilon.$$

Show that the error in the numerical integration rule applied to integrate f is bounded by

$$|\int_0^1 f(x)dx - \sum_{i=1}^{n} w_i f(x_i)| \le \Omega\epsilon$$

where

$$\Omega = 1 + \sum_{i=1}^{n} |w_i|.$$

Solution: Choose a polynomial p which approximates f within ϵ on the interval $[0,1]$. Since the rule is exact for p, we have

$$\begin{aligned} |\int_0^1 f(x)dx - \sum_{i=1}^{n} w_i f(x_i)| &= |\int_0^1 f(x)-p(x)dx + \sum_{i=1}^{n} w_i(p(x_i)-f(x_i))| \\ &\le \int_0^1 |f(x)-p(x)|dx + \sum_{i=1}^{n} |w_i||p(x_i)-f(x_i)| \\ &\le \Omega\epsilon. \end{aligned}$$

(48) **Problem 2** (a) Write out the Lagrange form of the quadratic polynomial $p(x)$ interpolating values f_1, f_2 and f_3 at points x_1, x_2 and x_3.
Solution:

$$\begin{aligned} p(x) &= \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} f_1 + \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} f_2 + \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} f_3 \\ &= \varphi_1(x)f_1 + \varphi_2(x)f_2 + \varphi_3(x)f_3 \end{aligned}$$

(b) Give a formula for the error $f(x)-p(x)$ in terms of f and the x_i's.
Solution:

$$f(x)-p(x) = \frac{f^{(3)}(\zeta)}{3!}(x-x_1)(x-x_2)(x-x_3)$$

(c) Assume IEEE standard floating-point arithmetic with machine precision $\epsilon < 1/200$. Assume that the values f_i are nonzero. Show that evaluating p at any point x gives the exact value at x of the quadratic polynomial $\hat{p}$ which interpolates values $\hat{f}_i$ satisfying

$$\frac{|f_i - \hat{f}_i|}{|f_i|} \le 40\epsilon,$$

at points x_1, x_2 and x_3. You may use the fact that

$$\prod_{i=1}^{n}(1+\delta_i)^{\sigma_i} = 1+\Delta$$

where $|\Delta| \le n\epsilon/(1-n\epsilon)$ if each $\delta_i \le \epsilon$, each $\sigma_i = \pm 1$, and $n\epsilon < 1$.
Solution: In floating-point arithmetic, the ith operation commits relative error δ_i bounded by the machine precision ϵ, so

$$\begin{aligned}
\mathrm{fl}(p(x)) &= ((\frac{(x-x_2)(1+\delta_1)(x-x_3)(1+\delta_2)(1+\delta_3)}{(x_1-x_2)(1+\delta_4)(x_1-x_3)(1+\delta_5)(1+\delta_6)}(1+\delta_7)f_1(1+\delta_8) \\
&\quad +\frac{(x-x_1)(1+\delta_9)(x-x_3)(1+\delta_{10})(1+\delta_{11})}{(x_2-x_1)(1+\delta_{12})(x_2-x_3)(1+\delta_{13})(1+\delta_{14})}(1+\delta_{15})f_2(1+\delta_{16}))(1+\delta_{17}) \\
&\quad +\frac{(x-x_1)(1+\delta_{18})(x-x_2)(1+\delta_{19})(1+\delta_{20})}{(x_3-x_1)(1+\delta_{21})(x_3-x_2)(1+\delta_{22})(1+\delta_{23})}(1+\delta_{24})f_3(1+\delta_{25}))(1+\delta_{26}) \\
&= \varphi_1(x)f_1(1+\Delta_1)+\varphi_2(x)f_2(1+\Delta_2)+\varphi_3(x)f_3(1+\Delta_3)
\end{aligned}$$

where each $|\Delta_i| \le 10\epsilon/(1-10\epsilon) \le 20\epsilon$. Defining $\hat{f}_i = f_i(1+\Delta_i)$ gives the result (with a factor of 2 to spare).

㊾ **Problem 3** (a) Find a numerical integration rule of the form

$$\int_0^3 f(x)dx = af(0)+bf(1)+cf(2)$$

which is exact whenever f is a polynomial of degree 2. Note that the upper limit of integration is 3, not 2.
Solution: If $f(x)$ is a polynomial of degree 2, then its quadratic interpolant p at the points 0, 1 and 2 is exactly equal to f. Thus

$$\int_0^3 f(x)dx = \int_0^3 \varphi_1(x)f(0)+\varphi_2(x)f(1)+\varphi_3(x)f(2)dx = af(0)+bf(1)+cf(2)$$

where

$$\begin{aligned}
a &= \int_0^3 \varphi_1(x)dx = \int_0^3 \frac{(x-1)(x-2)}{(0-1)(0-2)}dx = \frac{1}{2}(\frac{1}{3}x^3 - 3\frac{1}{2}x^2 + 2x)|_0^3 = \frac{3}{4} \\
b &= \int_0^3 \varphi_2(x)dx = \int_0^3 \frac{(x-0)(x-2)}{(1-0)(1-2)}dx = \frac{1}{-1}(\frac{1}{3}x^3 - 2\frac{1}{2}x^2)|_0^3 = 0 \\
c &= \int_0^3 \varphi_3(x)dx = \int_0^3 \frac{(x-0)(x-1)}{(2-0)(2-1)}dx = \frac{1}{2}(\frac{1}{3}x^3 - \frac{1}{2}x^2)|_0^3 = \frac{9}{4}
\end{aligned}$$

Note that $a+b+c=3$ as a check: the constant function 1 integrates to 3 as it should. Since this rule integrates quadratics exactly, its error must be bounded by

$$|\int_0^3 f(x)dx - af(0) - bf(1) - cf(2)| \le CM_3$$

for some constant C, whenever $|f^{(3)}(x)| \le M_3$ for all x.

(b) Assume we know the a, b and c from part (a). Find weights w_0, w_1 and w_2 such that the absolute error $E(h)$ in the approximation

$$\int_0^{3h} g(x)dx = \sum_{i=0}^{2} w_i g(ih) + E(h)$$

is of order $E(h) = O(h^4)$.
Solution: Let $f(t) = g(ht)$ where $0 \le t \le 3$. Then we know from part (a) that

$$\int_0^3 f(t)dt = af(0) + bf(1) + cf(2) + Cf^{(3)}(\zeta).$$

On the other hand,

$$\int_0^3 f(t)dt = \int_0^3 g(ht)dt = \frac{1}{h}\int_0^{3h} g(x)dx$$

and

$$f'(t) = hg'(ht), \qquad f''(t) = h^2 g''(ht), \qquad f'''(t) = h^3 g'''(ht).$$

Thus multiplying through by h gives

$$\int_0^{3h} g(x)dx = hag(0) + hbg(h) + hcg(2h) + O(Ch^4 M_3)$$

and

$$w_0 = ha = 3h/4, \qquad w_1 = hb = 0, \qquad w_2 = hc = 9h/4.$$

㊿

Extra Credit Problem 4 Let $p(x)$ be the polynomial of degree $n-1$ which interpolates $f(x)$ at n points x_i. Show that the derivatives satisfy

$$f'(x_i) - p'(x_i) = \frac{1}{n!} f^{(n)}(\xi) \prod_{j \ne i} (x_i - x_j)$$

(for some unknown point ξ in the interval $[\min_i x_i, \max_i x_i]$) at each interpolation point x_i.
Solution: We know that the error in polynomial interpolation is given by

$$f(x) - p(x) = \frac{f^{(n)}(\zeta_x)}{n!}(x - x_1)(x - x_2)\cdots(x - x_n)$$

where ζ_x depends continuously on x. Thus the definition of the derivative gives

$$\begin{aligned} f'(x_i) - p'(x_i) &= \lim_{h \to 0} \frac{f(x_i + h) - f(x_i) - p(x_i + h) + p(x_i)}{h} \\ &= \lim_{h \to 0} \frac{f^{(n)}(\zeta_{x_i + h})}{n!}(x_i + h - x_1)\cdots\frac{(x_i + h - x_i)}{h}\cdots(x_i + h - x_n) \\ &\to \frac{1}{n!} f^{(n)}(\zeta_{x_i}) \prod_{j \ne i}(x_i - x_j) \end{aligned}$$

as $h \to 0$.

Final Exam Solutions

(51) 1(a) $P = I - uu^T$ where $u = v/\|v\|_2$, so

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{21}\begin{bmatrix} 16 & 8 & 4 \\ 8 & 4 & 2 \\ 4 & 2 & 1 \end{bmatrix}$$

$$= \frac{1}{21}\begin{bmatrix} 5 & -8 & -4 \\ -8 & 17 & -2 \\ -4 & -2 & 20 \end{bmatrix}$$

(b) Since the columns of A are orthogonal to v, we can write

$$b = Pb + (I-P)b \qquad (I-P)b \perp \text{range}(A)$$

and

$$\|Ax-b\|_2^2 = \|Ax - Pb\|_2^2 + \|(I-P)b\|_2^2$$

Since

$$Pb = \frac{1}{21}\begin{bmatrix} 5 & -8 & -4 \\ -8 & 17 & -2 \\ -4 & -2 & 20 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{21}\begin{bmatrix} 5 \\ -8 \\ -4 \end{bmatrix}$$

we should solve

$$Ax = x_1\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \frac{1}{21}\begin{bmatrix} 5 \\ -8 \\ -4 \end{bmatrix}$$

to get

$$x_1 = -\frac{2}{21} \qquad x_2 = \frac{-5}{21}.$$

(c) Since $(I-P)c \to 0$ as $\theta \to 0$ (it is the part of c perpendicular to E), the general principle that computing zero output from nonzero input is bad guarantees that $\kappa \to 0$ as $\theta \to 0$. Explicitly,

$$\kappa(I-P,\, c) = \frac{\|I-P\|\,\|c\|}{\|(I-P)\,c\|} = \frac{1}{|\sin\theta|}\,.$$

(52) 2(a) $x_1 x_2 = 1$ and $x_1^2 = x_2^2 \implies x_1 = \pm 1 = x_2$. Thus $(x_1, x_2) = \pm(1,1)$.

(b) Newton says $f'(x^{(k)})\,(x^{(k+1)} - x^{(k)}) = -f(x^{(k)})$. Here

$$f(x) = \begin{bmatrix} x_1 x_2 - 1 \\ \frac{1}{2}(x_1^2 - x_2^2) \end{bmatrix} \qquad f'(x) = \begin{bmatrix} x_2 & x_1 \\ x_1 & -x_2 \end{bmatrix}$$

So

$$f'(x)^{-1} = \frac{-1}{r^2}\begin{bmatrix} -x_2 & -x_1 \\ -x_1 & x_2 \end{bmatrix} = \frac{1}{r^2}\begin{bmatrix} x_2 & x_1 \\ x_1 & -x_2 \end{bmatrix}$$

where $r^2 = x_1^2 + x_2^2$. Hence

$$x^{(k+1)} = x^{(k)} - \frac{1}{r^2}\begin{bmatrix} x_2 & x_1 \\ x_1 & -x_2 \end{bmatrix}\begin{bmatrix} x_1 x_2 - 1 \\ \frac{1}{2}(x_1^2 - x_2^2) \end{bmatrix}$$

$$= x^{(k)} - \frac{1}{r^2}\begin{bmatrix} x_1 x_2^2 + \frac{1}{2}x_1^3 - \frac{1}{2}x_1 x_2^2 - x_2 \\ x_1^2 x_2 - \frac{1}{2}x_2 x_1^2 + \frac{1}{2}x_2^3 - x_1 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \frac{1}{r^2}\begin{bmatrix} \frac{1}{2}r^2 x_1 - x_2 \\ \frac{1}{2}r^2 x_2 - x_1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{1}{r^2}\begin{bmatrix} x_2 \\ x_1 \end{bmatrix}.$$

This is not $\varphi(x)$. However, $\varphi(x) = x$ when $x_1 = x_2 = \pm 1$ so it has the correct fixed point.

(c) $x^{(0)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad x^{(1)} = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

$$x^{(2)} = \frac{1}{2}\begin{bmatrix} 1 \\ 1/2 \end{bmatrix} + \frac{4}{5}\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 18/20 \\ 21/20 \end{bmatrix}$$

(d) $e^{(0)} = 1$ (in the ∞-norm) for $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$e^{(1)} = 1/2, \quad e^{(2)} = 1/10.$$

In the 1-norm, $e^{(0)} = 1$, $e^{(1)} = 1/2$, $e^{(2)} = 3/20$. looks like a factor of 3 or so per step.

Definition: If

$$\|e^{(k+1)}\| = C\,\|e^{(k)}\|^p$$

(where $C < 1$ for $p = 1$) then the error has order p. Here, we estimate $1/2 \geq C \geq \frac{1}{3}$ and $p = 1$ (though we don't really have enough evidence). The linear convergence ($p = 1$) is typical for fixed point methods which are not Newton.

④

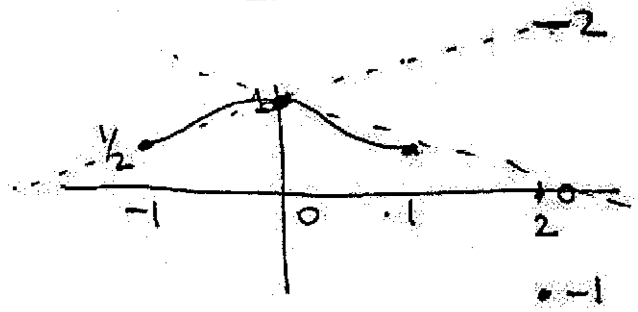
(53) 3(a) Aitken-Neville says

$$P_{1j} = f_j, \quad P_{2j} = \frac{(t-t_j)P_{1\,j+1} - (t-t_{j+1})P_{1j}}{t_{j+1}-t_j},$$

$$P_{3j} = \frac{(t-t_j)P_{2\,j+1} - (t-t_{j+2})P_{2j}}{t_{j+2}-t_j}.$$

Here $t = 2$ so we have

	t_j	P_{1j}	P_{2j}	P_{3j}
$j=0$	-1	$1/2$	$\frac{3\cdot 1 - 2\cdot 1/2}{1} = 2$ ✓	$\frac{3\cdot 0 - 1\cdot 2}{2} = -1$
1	0	1	$\frac{2\cdot 1/2 - 1\cdot 1}{1} = 0$ ✓	
2	1	$1/2$		



<u>Check by Lagrange</u>:

$$P(t) = \frac{(t-0)(t+1)}{(1-0)(1+1)}\,1/2 + \frac{(t-1)(t+1)}{(0-1)(0+1)}\cdot 1 + \frac{(t-0)(t-1)}{(-1-0)(-1-1)}\,1/2$$

$$= \tfrac{1}{4}t(t+1) + 1 - t^2 - \tfrac{1}{4}t(1-t)$$

$$P(2) = \tfrac{1}{4}\cdot 2\cdot 3 + 1 - 4 - \tfrac{1}{4}\cdot 2\cdot(-1) = \boxed{-1}$$ ✓

⑤

(b) For a quadratic interpolant,

$$e(x) = \frac{|f(x)-P(x)|}{|f(x)|} = \frac{|\frac{1}{3!}(x+1)x(x-1)|\,|f'''(\xi)|}{1/1+x^2}.$$

Since $f' = -1(1+x^2)^{-2}\cdot 2x$

$$f'' = +2(1+x^2)^{-3}\cdot(2x)^2 - 2(1+x^2)^{-2}$$

$$|f'''| = |-6(1+x^2)^{-4}(2x)^3 + (1+x^2)^{-3}\cdot 32x \leftarrow |$$

$$= \left|\frac{-48x^3 + 32(1+x^2)x}{(1+x^2)^4}\right| = \frac{|32x - 16x^2|}{(1+x^2)^4}$$

$$= \frac{16}{(1+x^2)^4}|x(2-x)|. \leq \text{[illegible]}\cdot|x|(2+|x|)$$

$$\leq \text{[illegible]}\cdot(1+h)(3+h) \qquad \text{for } 1 \leq x \leq 1+h$$

Bound increases with h because <u>extrapolation</u> is bad. $|e(x)| \leq \frac{1}{6}(1+h)(3+h)(1+(1+h)^2)^{2}\,h(h+1)(h+2)$

(c)

$$f(x)-P(x) = \frac{(x-t_0)\ldots(x-t_n)}{(n+1)!}f^{(n+1)}(\xi)$$

The $(x-t_0)\cdots$ guarantees exact interpolation, the $f^{(n+1)}$ reproduces degree-n polynomials exactly. Then the $(n+1)!$ comes from applying degree-n interpolation to $f(x) = x^{n+1}$.

⑥

(54) 4 (a) Let $\|f-P\|_\infty \leq \varepsilon$ and degree $(P) \leq d$.

Then

$$\left|\int_0^1 f(x)\,dx - \sum_{j=1}^{p} w_j \cdot f(x_j)\right|$$

$$= \left|\int_0^1 f(x) - P(x)\,dx + \int_0^1 P(x)\cdot dx - \sum_{j=1}^{p} w_j \cdot P(x_j) + \sum_{j=1}^{p} w_j P(x_j) - w_j \cdot f(x_j)\right|$$

$$\leq \int_0^1 \varepsilon\,dx + 0 + \sum_{j=1}^{p} |w_j| \varepsilon = \Lambda \varepsilon.$$

Since $\sum_{j=1}^{p} w_j = 1 \leq \sum_{j=1}^{p} |w_j|$, $\Lambda \geq 2$ is the best possible and happens iff the weights $w_j \geq 0$.

(b) Taylor expansion says

$$f(x) = f(0) + x f'(0) + \dots + \frac{x^d}{d!} f^{(d)}(0) + \frac{x^{d+1}}{(d+1)!} f^{(d+1)}(\xi)$$

$$= P(x) + \frac{x^{d+1}}{(d+1)!} f^{(d+1)}(\xi) = P(x) + E(x)$$

where $P(x)$ is a polynomial of degree d and

$$|E(x)| \leq \frac{1}{(d+1)!} \|f^{(d+1)}\|_\infty = \varepsilon.$$

(c) With p equidistant nodes we can get $d = p-1$ (or $d = p$ if p is even). Newton-Coles.

(d) With Gauss-Legendre nodes we can get $d = 2p-1$.

(55) 5. (a) Let $g(t) = f(t/h) = f(x)$ so

$$\int_0^h g(t)\,dt = \int_0^h f(t/h)\,dt = h\int_0^1 f(x)\,dx$$

$$= h\left[\sum_{j=1}^{p} w_j f(x_j) + C f^{(q)}(\xi)\right]$$

$$= \underbrace{h \sum_{j=1}^{p} w_j}_{W_j} g(\underbrace{h x_j}_{t_j}) + h^{q+1} g^{(q)}(h\xi).$$

(b) $S_{(J-1)n+j} = (J-1)h + h t_j$ may repeat!

$$W_{(J-1)n+j} = h W_j \quad \begin{cases} 0 \le J \le n \\ 1 \le j \le p \end{cases}$$

(c) Trapezoidal rule

$$s_j = a + (j-1)h \qquad 1 \le j \le n+1$$

$$w_j = \begin{cases} 1/2 & j = 1 \text{ or } n+1 \\ 1 & \text{otherwise} \end{cases}$$

There are really $2n$ nodes and weights but lots of them overlap.

Midterm 1 Solutions

(60) 1(a) Generally $\kappa_f = \|f'(x)\|\|x\|/\|f(x)\|$, so we need

$$f'(x) = \begin{bmatrix} r_1 & r_2 \\ r_1\theta + r\theta_1 & r_2\theta + r\theta_2 \end{bmatrix}$$

where subscripts denote partial derivatives. Differentiating $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$ gives

$$1 = r_1\cos\theta - r\theta_1\sin\theta$$
$$0 = r_2\cos\theta - r\theta_2\sin\theta$$

and

$$0 = r_1\sin\theta + r\theta_1\cos\theta$$
$$1 = r_2\sin\theta + r\theta_2\cos\theta$$

or

$$\begin{bmatrix} c & -rs \\ s & rc \end{bmatrix}\begin{bmatrix} r_1 & r_2 \\ \theta_1 & \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where

$$c = \cos\theta,\ s = \sin\theta.$$

Note that we have rederived part of the Inverse Function Theorem. Solving gives

$$\begin{bmatrix} r_1 & r_2 \\ \theta_1 & \theta_2 \end{bmatrix} = \frac{1}{r}\begin{bmatrix} rc & rs \\ -s & c \end{bmatrix} = \begin{bmatrix} c & s \\ -\frac{s}{r} & \frac{c}{r} \end{bmatrix},$$

So

$$f'(x) = \begin{bmatrix} c & s \\ c\theta - s & s\theta + c \end{bmatrix}.$$

In the ∞-norm we have

$$\|f'(x)\| = \text{max row } |\text{sum}|$$

$$= \max\left(|c|+|s| , |c\theta - s| + |s\theta + c| \right)$$

$$\leq \max(2, 2\pi+2) \quad \text{since } |\theta| \leq \pi.$$

$$\leq 2\pi + 2.$$

Moreover

$$\|x\|_\infty = 1$$

$$\|f(x)\|_\infty = \|(\sqrt{2}, \sqrt{2}\cdot\tfrac{\pi}{4})^T\|_\infty = \frac{\pi}{2\sqrt{2}} \geq 1$$

So

$$\kappa_f \leq 2\pi + 2 \leq 10.$$

The problem is well-conditioned.

③

(b) Since $\cdot + \sqrt{\ }$ deliver exact results correctly rounded,

$$\tilde{f}_1(x) = \sqrt{(x_1^2(1+\varepsilon_1) + x_2^2(1+\varepsilon_2))(1+\varepsilon_3)}\,(1+\varepsilon_4)$$

$$\doteq \sqrt{(x_1(1+2\varepsilon_5))^2 + (x_2(1+2\varepsilon_6))^2}$$

where $|\varepsilon_j| \le$ eps. Here we have ignored $O(\varepsilon^2)$ terms. Now

$$|x_1(1+2\varepsilon_5) - x_1| \le 2 \cdot \text{eps} \cdot |x_1|$$

and similarly for x_2, so $\tilde{f}_1$ is backward stable with $\sigma_B \le 2$.

(c) From (b), $\tilde{f}_1(x) - f_1(x) = f_1(\tilde{x}) - f_1(x)$, so

$$\frac{|\tilde{f}_1(x) - f_1(x)|}{|f_1(x)|} \le \kappa_f \frac{\|\tilde{x} - x\|}{\|x\|} \le 2\kappa_f \cdot \text{eps}$$

and $\tilde{f}_1$ is forward stable with $\sigma \le 2$.

④

(61) 2 (a)) First swap rows 1 and 3 of A with

$$P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

to get

$$A_1 = \begin{bmatrix} 10^6 & 0 & -3 \\ 1 & 0 & 3 \\ 0 & 2 & 3 \end{bmatrix} = P_1 A$$

Apply

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -10^{-6} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

to get

$$L_1 P_1 A = \begin{bmatrix} 10^6 & 0 & -3 \\ 0 & 0 & 3 + 10^{-6} \cdot 3 \\ 0 & 2 & 3 \end{bmatrix}$$

Apply

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = P_2^{-1}$$

to get

$$P_2 L_1 P_2^{-1} P_2 P_1 A = \begin{bmatrix} 10^6 & 0 & -3 \\ 0 & 2 & 3 \\ 0 & 0 & 3(1+10^{-6}) \end{bmatrix} = R$$

or $PA = LR$

where

$$P = P_2 P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$L = P_2 L_1^{-1} P_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 10^{-6} & 0 & 1 \end{bmatrix}$$

Check: $PA = LR$ ✓

(b) $K = \dfrac{\|R\| \, \|x\|}{\|Rx\|} = \dfrac{(10^6+3)\cdot 1}{3(1+10^{-6})} \approx \frac{1}{3}\cdot 10^6$ big!

This matrix-vector product is ill-conditioned.

(62) 3 (a) $P = I - \frac{1}{\|v\|_2^2} vv^T$ (6)

$$= \frac{1}{21}\begin{bmatrix} 20 & -2 & -4 \\ -2 & 17 & -8 \\ -4 & -8 & 5 \end{bmatrix}$$

(b) $$z = Pb = \frac{1}{21}\begin{bmatrix} -4 \\ -8 \\ 5 \end{bmatrix}$$

(c) Observe that the columns of A form a basis for E since $A_j^T v = 0$. Thus

$$Ax = z$$

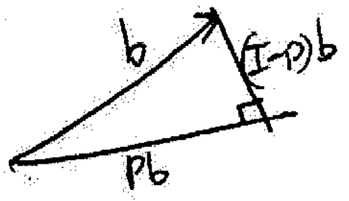
or

$$\begin{bmatrix} 2 & 0 \\ -1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4/21 \\ -8/21 \\ 5/21 \end{bmatrix}.$$

Hence

$$x_1 = -8/42$$
$$x_2 = -10/42.$$

(d) $$K = \frac{\|I-P\|_2 \|b\|_2}{\|(I-P)b\|_2} = \frac{1}{\sin\theta}.$$



Midterm 2 Solutions

1(a) $c = (1,1)^T$ by inspection.

(b) Newton says to solve

$$f'(x^{(k)})\, e = -f(x^{(k)})$$

and then update $x^{(k+1)} = x^{(k)} + e$.
Here,

$$f'(x) = \begin{bmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{bmatrix}$$

so

$$f'(x)^{-1} = \frac{1}{x_1^2+x_2^2} \begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix}$$

and

$$e = -f'(x)^{-1} f(x)$$

$$\underset{r^2 =}{=} \overset{-}{\frac{-1}{x_1^2+x_2^2}} \begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 \\ x_1x_2 - 1 \end{bmatrix}$$

$$= \frac{-1}{r^2} \begin{bmatrix} \frac{1}{2}x_1^3 - \frac{1}{2}x_1x_2^2 + x_1x_2^2 - x_2 \\ -\frac{1}{2}x_1^2x_2 + \frac{1}{2}x_2^3 + x_1^2x_2 - x_1 \end{bmatrix}$$

$$e = -\frac{1}{r^2}\begin{bmatrix} \frac{1}{2}x_1 r^2 - x_2 \\ \frac{1}{2}x_2 r^2 - x_1 \end{bmatrix} = -\frac{1}{2}x + \frac{1}{r^2}\begin{bmatrix} x_2 \\ x_1 \end{bmatrix}.$$

Thus the update gives

$$x^{(k+1)} = x^{(k)} + e = \frac{1}{2}x + \frac{1}{r^2}\begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

and the given iteration is Newton.

(c) $x^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$x^{(1)} = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

$$x^{(2)} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} + \frac{1}{5/4}\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{21}{20} \\ \frac{9}{10} \end{bmatrix}$$

The errors are (in maximum norm)

$$e_\infty^{(0)} = 1, \quad e^{(1)} = \frac{1}{2}, \quad e^{(2)} = \frac{1}{10}$$

and display the beginnings of quadratic convergence $e^{(n+1)}/c = \rho\,(e^n/c)^2$ with

$$\rho = \frac{1}{2} \quad \text{or} \quad \rho = 4/10.$$

(67) ③

2 (a) Lagrange:

$$P(x) = 1 \cdot \frac{(x-2)(x-3)}{(1-2)(1-3)} + 8\,\frac{(x-1)(x-3)}{(2-1)(2-3)} + 27\,\frac{(x-1)(x-2)}{(3-1)(3-2)}$$

Newton: First compute divided differences

x	f	[,]f	
1	1	7	$6 = (7-19)/(-2)$
2	8	19	
3	27		

alternately, $[1]f = f(1) = 1$

$$[2,1]f = \frac{f(2)-f(1)}{2-1} = 7$$

$$[3,2,1]f = \frac{f(3)-2f(2)+f(1)}{(2-1)(3-1)} = 6$$

so

$$P(x) = 1 + 7(x-1) + 6(x-1)(x-2).$$

<u>Check</u>: $P(1) = 1$

$P(2) = 8$

$P(3) = 1 + 14 + 12 = 27$ ✓

(b) From general formula: note $f(x) \geq 1$ on $[1,3]$ so

$$|e(x)| = \left|\frac{f^{(3)}(\xi)}{6}(x-1)(x-2)(x-3)\right| \leq 4$$

④

(c) while on $[1.9, 2.1]$

$$|x-1|\,|x-2|\,|x-3| \leq (1.1)(0.1)(1.1) = 0.121$$

much smaller because polynomial interpolation works well near center of interval containing nodes.

(d)

$$f(x) - P(x) = \underset{①}{\frac{1}{(n+1)!}}\;\underset{②}{f^{(n+1)}(\xi)}\;\underset{③}{(x-t_0)\dots(x-t_n)}$$

Factor ② must vanish when f is a degree-n polynomial.

Factor ③ makes the error vanish at the nodes.

(68) 3.~~(a) Clearly $B_j^n(t) \geq 0$, while then $\sum_{j=0}^{n} B_j^n(t) = 1$ implies $B_j^n(t) \leq 1$. This is useful because a polynomial $P(t) = \sum_{j=0}^{n} b_j B_j^n(t)$ is a convex combination of coefficients b_j at each $t \in [0,1]$.~~

(b) First observe that

$$B_0 = (1-t)^3 \quad B_1 = 3(1-t)^2 t$$

$$B_2 = 3(1-t)t^2 \quad B_3 = t^3$$

so

$$B_0' = -3(1-t)^2 \quad B_1' = 3\left[-2(1-t)t + (1-t)^2\right]$$

$$B_2' = 3\left[-t^2 + 2(1-t)t\right] \quad B_3' = 3t^2$$

So

$$P(0) = b_0 \qquad P'(0) = 3(b_1 - b_0)$$

$$P(1) = b_3 \qquad P'(1) = 3(b_3 - b_2).$$

Thus $b_0 = f_0$, $b_3 = f_1$, and

$$3(b_1 - f_0) = f_0' \Rightarrow b_1 = f_0 + \tfrac{1}{3} f_0'$$

$$3(f_1 - b_2) = f_1' \qquad b_2 = f_1 - \tfrac{1}{3} f_1'$$

Hence

$$\boxed{\begin{aligned} P(t) = {} & f_0 B_0 + \left(f_0 + \tfrac{1}{3} f_0'\right) B_1 \\ & + \left(f_1 - \tfrac{1}{3} f_1'\right) B_2 + f_1 B_3 . \end{aligned}}$$