

1 Let

$$f(x, y) = \begin{bmatrix} x^2/4 + y^2/9 - 1 \\ x - y - 1 \end{bmatrix}.$$

(a) Define quadratic convergence of a sequence of vectors x_n to a limit x . (b) Compute the Jacobian matrix $Df(x, y)$. (c) Determine where $Df(x, y)$ is invertible and compute $Df(x, y)^{-1}$ when it exists. (d) Write down Newton's method for solving $f(x, y) = 0$. (e) Start with $x_0 = (2, 0)^T$ and compute the first two approximations x_1 and x_2 generated by Newton's method. (f) Explain why your results demonstrate quadratic convergence.

2 Consider the iteration

$$x_{n+1} = \frac{x_n^3 + 3ax_n}{3x_n^2 + a}.$$

(a) What is it intended to compute? (b) Given $a = 2$ and $x_0 = 1$, compute x_1 and x_2 . (c) Define and determine the order of convergence of this iteration.

3 Find the QR factorization of

$$A = \begin{bmatrix} 5 & 0 \\ 3 & 5 \\ 4 & 10 \end{bmatrix} = Q_1 Q_2 Q_3 R = \frac{1}{5} P_1 \frac{1}{\sqrt{2}} P_2 \frac{1}{\sqrt{129}} P_3 R.$$

You don't need to multiply together the matrices P_i .

4 (a) Derive a numerical integration formula

$$\int_0^1 f(x) dx = w_0 f(0) + w_1 f(1) + w_2 f(2)$$

which is exact for polynomials of as high degree d as possible, and determine the maximal degree d . (b) Without any additional work, determine an equally accurate rule of the form

$$\int_0^1 f(x) dx = u_0 f(-1) + u_1 f(0) + u_2 f(1).$$

(c) Show that

$$\int_0^h f(x) dx = h(w_0 f(0) + w_1 f(h) + w_2 f(2h)) + O(h^{d+1})$$

as $h \rightarrow 0$. (d) Use (a), (b), and (c) to build a quadrature formula with error $O(h^d)$ on an arbitrary interval $[a, b]$ divided into $n > 1$ subintervals of length $h = (b - a)/n$.

5 (a) Write down the Newton and Lagrange forms of the quadratic interpolant $p(x)$ to a function f at three points a , b and c . **(b)** Give a formula for the error $p(x) - f(x)$ if f is a nice function with all derivatives bounded. Explain why your error formula makes sense in terms of dimensions, zeroes and the derivatives which appear versus the degree of polynomial used. **(c)** Specialize to $f(x) = R - 1/x$ and evaluate the coefficients in the Newton representation of $p(x)$. **(d)** Use (c) to express p in the power form $p(x) = q_0 + q_1x + q_2x^2$. **(e)** How would you use the formula of (d) to derive an iterative method for finding $1/R$?

6 Suppose A is a square invertible matrix. **(a)** Define the condition number $\kappa(A)$. **(b)** Suppose E is a matrix the same size as A and

$$\kappa(A) \frac{\|E\|}{\|A\|} \leq \epsilon \leq \frac{1}{2}.$$

Show that $A + E$ is invertible. **(c)** Show that

$$\frac{\|(A + E)^{-1} - A^{-1}\|}{\|A^{-1}\|} \leq 2\epsilon.$$

7 Given an approximate solution y to the linear system $Ax = b$ with a square invertible matrix A , let $r = b - Ay$ be the residual of y . **(a)** Show that if y satisfies a perturbed linear system $(A + E)y = b$ then the perturbation E must satisfy

$$\frac{\|E\|}{\|A\|} \geq \frac{\|r\|}{\|A\|\|y\|}.$$

(b) Show that there is a matrix E such that $(A + E)y = b$ with the norm of E satisfying

$$\frac{\|E\|}{\|A\|} = \frac{\|r\|}{\|A\|\|y\|}.$$

(Hint: Try a rank-one matrix $E = \alpha r y^T$ for some well-chosen scalar α .) **(c)** Define the backward relative error in an approximate solution y of $Ax = b$. **(d)** Show that an approximate solution y of $Ax = b$ has backward relative error $O(\epsilon)$ if and only if it has a residual r satisfying

$$\frac{\|r\|}{\|A\|\|y\|} = O(\epsilon).$$

8 Prove that any model of floating-point arithmetic which requires that the floating-point result of the multiplication $x * y$ be given by the exact result correctly rounded satisfies the relative error bound

$$\frac{|x * y - \text{fl}(x * y)|}{|x * y|} \leq \epsilon$$

as long as no overflow or underflow occurs and $x * y \neq 0$.

(9) Given 10 values $f(x_j)$ at distinct points $0 < x_j < \pi$, show that there are unique coefficients c_j such that

$$f(x_j) = c_1 \sin(x_j) + \cdots + c_{10} \sin(10x_j)$$

for $j = 1 : 10$. (Hint: apply linear algebra, then differentiate something $2n$ times and let $n \rightarrow \infty$.)

(10) Suppose a quadrature rule

$$\int_0^1 f(x) dx = \sum_{j=1}^n W_j f(X_j) + E_n(f)$$

has error term $E_n(f) = C_n f^{(2n)}(\xi)$ for some unknown point $\xi \in [0, 1]$. (a) Find points and weights of a quadrature rule on $[0, h]$ with

$$\int_0^h f(x) dx = \sum_{j=1}^n w_j f(x_j) + O(h^{2n+1} f^{(2n)})$$

as $h \rightarrow 0$. (b) Describe a quadrature rule on an arbitrary interval $[a, b]$ with

$$\int_a^b f(x) dx = \sum_{j=1}^N u_j f(z_j) + O(h^{2n})$$

where $N = np$ and $h = (b - a)/p$.

(11) Find the QR decomposition of

$$A = \begin{bmatrix} 2 & 1 \\ -2 & 2 \\ 4 & 5 \end{bmatrix}$$

and check that $QR = A$. Find an orthonormal basis for the range of A and express each orthonormal basis vector as a linear combination of the columns of A .

(12) Define the condition number $\kappa(A)$ and use it to bound the change in the solution x of $Ax = b$ when A is changed to $A + E$ with E small.

(13) Calculate $f[0, 1, 2, 3]$ where $f(x) = 1/(1 + x)$.

(14) Let $F(x) = \cos(x)$. Derive a bound for the maximum error on $[-1, 1]$ when f is approximated at a point x by (a) a 3-term Taylor expansion about 0, and (b) quadratic interpolation from the nearest points to x in an equidistant n -point grid on $[-1, 1]$.

(15) Let

$$f(x, y) = \begin{bmatrix} x^2 + y^2 - 1 \\ x + y - 1 \end{bmatrix}.$$

(a) Define quadratic convergence of a sequence of vectors x_n to a limit x . (b) Compute the Jacobian matrix $Df(x, y)$. (c) Determine where $Df(x, y)$ is invertible and compute $Df(x, y)^{-1}$ when it exists. (d) Write down Newton's method for solving $f(x, y) = 0$. (e) Start with $x_0 = (2, 0)^T$ and compute the first two approximations x_1 and x_2 generated by Newton's method. (f) Explain why your results demonstrate quadratic convergence.

(16) Let A be an n by n matrix. (a) Let X be another n by n matrix such that $AX = I$. Show that A is invertible and $XA = I$. (b) Define the condition number $\kappa(A)$ and derive the bound

$$\|X - A^{-1}\| \leq \kappa(A) \frac{\|AX - I\|}{\|A\|}$$

for any operator norm. (c) Let

$$A = \begin{bmatrix} 9999 & 9998 \\ 10000 & 9999 \end{bmatrix}.$$

Compute $\|A\|_1$ and A^{-1} . (d) Let

$$X = \begin{bmatrix} 9999.9999 & -9997.0001 \\ -10001 & 9998 \end{bmatrix}.$$

Compute $\|AX - I\|_1$ and $\|XA - I\|_1$. (e) Use (b), (c) and (d) to show that $\kappa(A) \geq 10^6$.

(17) Build orthogonal polynomials $p_n(x)$ on the interval $[a, b]$ by the three-term recurrence

$$p_{n+1}(x) = \alpha_n((x - \beta_n)p_n(x) - \gamma_n p_{n-1}(x))$$

from $p_0 = 1$ and $p_1 = cx + d$. Derive formulas for α_n , β_n and γ_n in terms of p_n and p_{n-1} which make the polynomials both orthogonal and normalized:

$$\int_a^b p_n(x)p_m(x) = \delta_{nm}.$$

(18) Consider the equation

$$f(x) = x - 1 - 1/x = 0.$$

(a) Find all roots of $f(x) = 0$. (b) For which roots will fixed point iteration converge from sufficiently accurate starting points? (c) For which roots will Newton's method converge from sufficiently accurate starting points?

(19) Suppose f is a nice smooth function and r is a point where $f(r) = f'(r) = 0$ but $f''(r) \neq 0$. Determine how fast the iteration

$$x_{n+1} = x_n - \frac{u(x_n)}{u'(x_n)}$$

converges near $x = r$ if u is defined by

$$u(x) = \frac{f(x)}{f'(x)}.$$

(20) Find the LU decomposition of

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -2 & 2 & -2 \\ 4 & 5 & 1 \end{bmatrix}$$

and check that $LU = A$.

(21) Consider the iteration

$$x_{n+1} = \frac{x_n^3 + 3ax_n}{3x_n^2 + a}.$$

(a) What is it intended to compute? (b) Given $a = 2$ and $x_0 = 1$, compute x_1 and x_2 . (c) Define and determine the order of convergence of this iteration.

(22) The secant method can be written in two algebraically equivalent forms

(a)

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

and (b)

$$x_{n+1} = \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}.$$

Which is better in floating-point arithmetic and why?

(23) The mean M and variance V of n numbers x_i are defined by

$$M = \frac{1}{n} \sum_{i=1}^n x_i, \quad V = \frac{1}{n} \sum_{i=1}^n (x_i - M)^2 = \left(\frac{1}{n} \sum_{i=1}^n x_i^2 \right) - M^2$$

where the two formulas for V are equivalent in exact arithmetic because

$$\sum_{i=1}^n (x_i - M) = 0.$$

Take $n = 2$ for simplicity, and show that in floating-point arithmetic with machine precision ϵ the two formulas satisfy different error bounds

$$|V - \text{fl} \left(\frac{1}{2} \sum_{i=1}^2 (x_i - M)^2 \right)| \leq 5\epsilon|V| + O(\epsilon^2)$$

and

$$|V - \text{fl} \left(\left(\frac{1}{2} \sum_{i=1}^2 x_i^2 \right) - M^2 \right)| \leq 2\epsilon(x_1^2 + x_2^2 + M^2) + O(\epsilon^2)$$

Which formula has a better guaranteed accuracy in floating-point arithmetic?

(24) Suppose that a numerical integration rule

$$\int_0^1 f(x) dx \approx \sum_{i=1}^n w_i f(x_i)$$

with points $x_i \in [0, 1]$ is exact for polynomials of degree d . Suppose that f can be approximated by a polynomial p of degree d to accuracy ϵ on the interval $[0, 1]$:

$$\max_{0 \leq x \leq 1} |f(x) - p(x)| \leq \epsilon.$$

Show that the error in the numerical integration rule applied to integrate f is bounded by

$$\left| \int_0^1 f(x) dx - \sum_{i=1}^n w_i f(x_i) \right| \leq \Omega \epsilon$$

where

$$\Omega = 1 + \sum_{i=1}^n |w_i|.$$

(25) Compute the Newton form of the cubic polynomial which interpolates the function $f(x) = |x - 1|$ at the points $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ and $x_4 = 0$.

(26) Suppose we define orthogonal polynomials with respect to a different inner product

$$f^*g = \int_{-1}^1 f(x)g(x) \frac{dx}{\sqrt{1-x^2}}.$$

(a) Compute the first three orthogonal polynomials $p_0(x) = 1$, $p_1(x) = x$, and $p_2(x)$. (Hint: put $x = \cos(t)$ to evaluate the integrals.)

(b) Find the points of the 2-point Gaussian quadrature formula for evaluating weighted integrals of the form

$$\int_{-1}^1 f(x) \frac{dx}{\sqrt{1-x^2}}.$$

(27) (a) Consider the `matlab` code

```
x=1; q=0;
while x>0
    x=x/2;
    q=q+1;
end;
```

In binary floating-point arithmetic, what does it evaluate?

(b) Consider the matlab code

```
x=1;
while 1 + x > 1
    x=x/2;
end;
```

In binary floating-point arithmetic, what does it evaluate?

(c) Consider the matlab code

```
function a = f(x,y,z)
n = length(x);
for k = 1:n-1
    y(k+1:n) = (y(k+1:n)-y(k:n-1)) ./ (x(k+1:n) - x(1:n-k));
end
a = y(n)*ones(size(z));
for k=n-1:-1:1
    a = (z-x(k)).*a + y(k);
end
```

In exact arithmetic, what would it evaluate?

(28) (a) Find the Lagrange form of the quadratic polynomial interpolating the function $f(x) = x^3$ at the points $x_1 = 1$, $x_2 = 2$, $x_3 = 3$.

(b) Find a numerical integration rule of the form

$$\int_0^3 f(x)dx = af(1) + bf(2) + cf(3)$$

which is exact whenever f is a polynomial of degree 2. Note that the lower limit of integration is 0, not 1.

(c) Give weights w_1 , w_2 and w_3 such that the error $E(h)$ in the approximation

$$\int_0^{3h} f(x)dx = \sum_{i=1}^3 w_i f(ih) + E(h)$$

is of order $E(h) = O(h^4)$.

(29) (a) Find the power form of the linear polynomial $p(x)$ interpolating the function $f(x) = x^2$ at the points $x_1 = 1$ and $x_2 = 2$.

(b) Show that the absolute error in evaluating $p(x)$ at an arbitrary point x by floating-point arithmetic with machine precision ϵ is bounded by

$$|p(x) - \text{fl}(p(x))| \leq 8(|x| + 1)\epsilon.$$

(c) Show that $\text{fl}(p(x)) = p(\hat{x})$ is equal to the exact value of the polynomial p at a point \hat{x} such that $|x - \hat{x}| \leq 5(1 + |x|)\epsilon$.

(30) Let k be an arbitrary positive integer and consider interpolating $\sin(kx)$ by a polynomial of degree 10 at 11 equispaced points on the interval $[0, 1]$. Give a bound for the error $|\sin(kx) - p(x)|$ as a function of k when x lies in the interval $[0.0, 0.1]$.

(31) Let

$$\varphi_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

be the i th Lagrange basis function for interpolation at n points x_1, \dots, x_n . Show that

$$\alpha_k(x) = (1 - 2\varphi'_k(x_k)(x - x_k))\varphi_k(x)^2$$

and

$$\beta_k(x) = (x - x_k)\varphi_k(x)^2$$

are basis functions for Hermite interpolation of a function f and its derivative f' at the points x_1, \dots, x_n .

(32) Suppose a function f can be evaluated with relative error bounded by $\delta < 1/2$. We approximate its derivative by the forward difference quotient

$$D_h f(x) = \frac{f(x+h) - f(x)}{h}.$$

(a) Use Taylor expansion to show that in exact arithmetic

$$|D_h f(x) - f'(x)| \leq \left| \frac{M_2}{2} h^2 \right|$$

whenever $|f''(x)| \leq M_2$ for all x .

(b) Show that in floating-point arithmetic with machine epsilon $\epsilon < 1/2$ the error in evaluating the approximation is bounded by $6(M_0 + M_1)(\epsilon + \delta)/h$ whenever $|f(x)| \leq M_0$ and $|f'(x)| \leq M_1$ for all x .

(c) Find h (as a function of $\epsilon + \delta$, M_0 , M_1 and M_2) which gives the best error bound on the approximation of $f'(x)$ by $\text{fl}(D_h f(x))$.

(33) Consider two sets of data

$$x_1 = 1, \quad x_2 = 2, \quad x_3 = 3, \quad y_1 = 1, \quad y_2 = 12, \quad y_3 = 13,$$

and

$$\hat{x}_1 = 3, \quad \hat{x}_2 = 2, \quad \hat{x}_3 = 1, \quad \hat{y}_1 = 13, \quad \hat{y}_2 = 12, \quad \hat{y}_3 = 1.$$

Prove or disprove: the same quadratic polynomial $p(x)$ interpolates both sets of data.

(34) Describe three techniques for proving the following theorem: given n distinct points $x_i \in R$ and n values $y_i \in R$, there is a unique polynomial p of degree $n - 1$ such that $p(x_i) = y_i$ for $i = 1 : n$.

(35) Suppose $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. Describe two methods for evaluating p at a given point x . Show that the two methods are equally accurate in floating-point arithmetic, but one is faster.

(36) Verify that $p(x) = 3+2(x-1)+4(x-1)(x+2)$ and $q(x) = 4x^2+6x-7$ both interpolate $q(x)$ at $x = 1, -2$ and 0 . Explain why this does not contradict the uniqueness of polynomial interpolation.

(37) The polynomial $p(x) = x^4 - x^3 + x^2 - x + 1$ has values 31, 5, 1, 1, 11, 61 at the points $-2, -1, 0, 1, 2, 3$. Find a degree-5 polynomial $q(x)$ with values 31, 5, 1, 1, 11, and 30 at the same points.

(38) Let $p(x)$ have degree n . What are the divided differences $p[x_1, x_2, \dots, x_{n+2}]$ for $n + 2$ distinct points x_i ? Prove it.

(39) Let $p(x)$ be the polynomial of degree $n - 1$ which interpolates $f(x)$ at n points x_i . Show that the derivatives satisfy

$$f'(x_i) - p'(x_i) = \frac{1}{n!} f^{(n)}(\xi) \prod_{j \neq i} (x_i - x_j)$$

at each interpolation points x_i .

(40) Let $|\delta_i| \leq \delta$ for $i = 1 : n$, where $n\delta < 1/2$.

(a) Show that

$$\prod_{j=1}^n (1 + \delta_j) = 1 + \Delta$$

where $|\Delta| \leq 2n\delta$.

(b) Show that

$$\prod_{j=1}^n (1 + \delta_j)^{-1} = 1 + \Delta$$

where $|\Delta| \leq n\delta/(1 - n\delta)$.

(41) Prove that the $n \times n$ matrix with 4's down the diagonal, -1 's adjacent to the diagonal and 0's elsewhere is invertible for any n .

(42) Suppose x and y are floating-point numbers and the floating-point subtraction $\text{fl}(x - y)$ evaluates to 0. Prove or give a counterexample: $x = y$.

(43) Define the machine precision ϵ for floating-point arithmetic. Prove that the model of floating-point arithmetic which requires that the floating-point result of the addition $x + y$ be given by the exact result correctly rounded satisfies the relative error bound

$$\frac{|x + y - \text{fl}(x + y)|}{|x + y|} \leq \epsilon$$

as long as no overflow or underflow occurs and $x + y \neq 0$.

(44) Derive a numerical integration formula

$$\int_0^1 f(x)dx = w_0f(0) + w_1f(1) + w_2f(2)$$

which is exact for polynomials of as high degree d as possible, and determine the maximal degree d . Without any additional work determine a comparable rule of the form

$$\int_0^1 f(x)dx = u_0f(0) + u_1f(1) + u_2f(-1).$$

(45) Show that for any function f on the interval $[-1, 1]$ and any nonzero s , there exists a unique function of the “exponential spline” form

$$p(t) = a + bt + c \cosh(st) + d \sinh(st)$$

such that

$$p(-1) = f(-1), \quad p(+1) = f(+1), \quad p'(-1) = f'(-1), \quad p'(+1) = f'(+1).$$

(46) Show that for any function f on the interval $[-1, 1]$ and any $s \in (0, \pi)$, there exists a unique function of the “trigonometric spline” form

$$p(t) = a + bt + c \cos(st) + d \sin(st)$$

such that

$$p(-1) = f(-1), \quad p(+1) = f(+1), \quad p'(-1) = f'(-1), \quad p'(+1) = f'(+1).$$

47. Suppose that a numerical integration rule

$$\int_0^1 f(x)dx \approx \sum_{i=1}^n w_i f(x_i)$$

with points $x_i \in [0, 1]$ is exact for polynomials of degree d . Suppose that a function f can be approximated by a polynomial p of degree d to accuracy ϵ on the interval $[0, 1]$:

$$\max_{0 \leq x \leq 1} |f(x) - p(x)| \leq \epsilon.$$

Show that the error in the numerical integration rule applied to integrate f is bounded by

$$\left| \int_0^1 f(x) dx - \sum_{i=1}^n w_i f(x_i) \right| \leq \Omega \epsilon$$

where

$$\Omega = 1 + \sum_{i=1}^n |w_i|.$$

48. **(a)** Write out the Lagrange form of the quadratic polynomial $p(x)$ interpolating values f_1, f_2 and f_3 at points x_1, x_2 and x_3 .

(b) Give a formula for the error $f(x) - p(x)$ in terms of f and the x_i 's.

(c) Assume IEEE standard floating-point arithmetic with machine precision $\epsilon < 1/200$. Assume that the values f_i are nonzero. Show that evaluating p at any point x gives the exact value at x of the quadratic polynomial \hat{p} which interpolates values \hat{f}_i satisfying

$$\frac{|f_i - \hat{f}_i|}{|f_i|} \leq 40\epsilon,$$

at points x_1, x_2 and x_3 . You may use the fact that

$$\prod_{i=1}^n (1 + \delta_i)^{\sigma_i} = 1 + \Delta$$

where $|\Delta| \leq n\epsilon/(1 - n\epsilon)$ if each $\delta_i \leq \epsilon$, each $\sigma_i = \pm 1$, and $n\epsilon < 1$.

49. **(a)** Find a numerical integration rule of the form

$$\int_0^3 f(x) dx = af(0) + bf(1) + cf(2)$$

which is exact whenever f is a polynomial of degree 2. Note that the upper limit of integration is 3, not 2.

(b) Assume we know the a, b and c from part (a). Find weights w_0, w_1 and w_2 such that the absolute error $E(h)$ in the approximation

$$\int_0^{3h} g(x) dx = \sum_{i=0}^2 w_i g(ih) + E(h)$$

is of order $E(h) = O(h^4)$.

50. Let $p(x)$ be the polynomial of degree $n - 1$ which interpolates $f(x)$ at n points x_i . Show that the derivatives satisfy

$$f'(x_i) - p'(x_i) = \frac{1}{n!} f^{(n)}(\xi) \prod_{j \neq i} (x_i - x_j)$$

(for some unknown point ξ in the interval $[\min_i x_i, \max_i x_i]$) at each interpolation point x_i .

51. (a) Find the matrix P which implements orthogonal projection onto the plane E consisting of all 3-vectors perpendicular to $v = (4, 2, 1)^T$.

(b) Find $x \in \mathbb{R}^2$ minimizing $\|Ax - b\|_2$ where

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 2 \\ 2 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

(c) Compute the 2-norm condition number for the problem of multiplying $I - P$ into an arbitrary vector $c \in \mathbb{R}^3$, in terms of the angle θ between c and E . Explain from general principles why the condition number must become infinite as $\theta \rightarrow 0$.

52. (a) Find all solutions $x = (x_1, x_2)^T$ of the 2×2 nonlinear system

$$f(x_1, x_2) = \left(x_1 x_2 - 1, \frac{1}{2}(x_1^2 - x_2^2) \right)^T = (0, 0)^T.$$

(b) Determine whether the fixed point iteration $x^{(k+1)} = \varphi(x^{(k)})$ with

$$\varphi(x) = \frac{1}{2}(x_2, x_1)^T + \frac{1}{x_1^2 + x_2^2}(x_1, x_2)^T$$

is the same as Newton's method for solving $f = 0$.

(c) Start from $x^{(0)} = (0, 1)^T$ and compute two iterates $x^{(1)}$ and $x^{(2)}$.

(d) Compute the errors in $x^{(1)}$ and $x^{(2)}$. Define, estimate and explain the rate of convergence.

53. Let $P(x)$ be the quadratic polynomial that interpolates $f(x) = 1/(1+x^2)$ at $x = 0, 1$ and -1 .

(a) Use the Aitken-Neville technique to evaluate P at $x = 2$.

(b) Bound the relative error $e(x) = |f(x) - P(x)|/|f(x)|$ on the interval $1 \leq x \leq 1 + h$ for any $h \geq 0$. Why does your bound increase as h increases?

(c) State the general formula which gives the error at a point t in the degree- n polynomial P which interpolates a smooth function f at $n + 1$ points t_j in $[a, b]$. Separate the error into a product of three factors and explain why two of them are inevitable.

54. Suppose that a numerical integration rule with nodes $x_j \in [0, 1]$, weights w_j , and error

$$E_p(f) = \int_0^1 f(x) dx - \sum_{j=1}^p w_j f(x_j)$$

is exact for polynomials of degree d : $E_p(f) = 0$ whenever f is a polynomial of degree d .

(a) Assume f can be approximated by a degree- d polynomial within error ϵ on the whole interval $[0, 1]$. Prove that $|E_p(f)| \leq \Lambda\epsilon$ where $\Lambda = 1 + \sum_{j=1}^p |w_j|$. What is the best possible value of Λ and when does it occur?

(b) Prove that the best possible error ϵ which we can hope to achieve in (a) is less than some constant times the maximum absolute value of the p th derivative $f^{(p)}$ over the interval $[0, 1]$.

(c) If the nodes x_j are equidistant on $[0, 1]$, with $x_1 = 0$ and $x_p = 1$, what is the biggest possible value of d ? What are the resulting rules called?

(d) If you get to choose the nodes x_j anywhere in $[0, 1]$, how big can you make d ? What are the resulting rules called?

55. Suppose that a numerical integration rule with nodes $x_j \in [0, 1]$ and weights w_j has an error of the form

$$\int_0^1 f(x)dx - \sum_{j=1}^p w_j f(x_j) = C f^{(q)}(\xi)$$

for some unknown ξ .

(a) Prove that the nodes $t_j = hx_j$ in the interval $[0, h]$ and the weights $u_j = hw_j$ satisfy the error estimate

$$\int_0^h g(t)dt - \sum_{j=1}^p u_j g(t_j) = Ch^{q+1}g^{(q)}(\xi)$$

for some unknown ξ .

(b) For arbitrary a and b and $N = np$, write down formulas for nodes $s_j \in [a, b]$ and weights v_j such that

$$\int_a^b g(s)ds - \sum_{j=1}^N v_j g(s_j) \leq C|b-a|h^q \max_{a \leq s \leq b} |g^{(q)}(s)|$$

where $h = (b-a)/n$.

(c) Find the weights v_j and nodes s_j on an arbitrary interval $[a, b]$ for the special case with $p = 2$ equidistant nodes $x_1 = 0$, $x_2 = 1$ and $q = 2$.

57. (a) Define the (normwise) condition number κ_f for the problem of evaluating a function $f : R^m \rightarrow R^n$ at input x .

(b) Give a general formula for κ_f .

(c) Define forward and backward stability of an algorithm $\tilde{f}(x)$ for evaluating f at x .

(d) Evaluate κ_f for

$$f(x) = (x_1^2 - x_2^2, 2x_1x_2)^T = (f_1(x), f_2(x))^T$$

at $x = (1, 1)^T$. Is this problem well-conditioned?

(e) Consider the evaluation of f (from part (d)) by multiplication and subtraction (in that order). Show that each component, \tilde{f}_1 and \tilde{f}_2 , is backward stable. Explain why we do not expect \tilde{f} as a whole to be backward stable at x .

(f) Use (e) to show that f is forward stable at x .

58. (a) Use Gaussian elimination to find a permutation matrix P , a unit lower triangular matrix L , and an upper triangular matrix R such that $PA = LR$ where

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}.$$

(b) Repeat with

$$A = \begin{bmatrix} 1 & 2 \\ 300 & 4 \end{bmatrix}.$$

(c) Find vectors u and v such that

$$L = \begin{bmatrix} 1 & 0 \\ 300 & 1 \end{bmatrix} = I + uv^T.$$

Express L^{-1} in terms of u and v .

(d) Compute the 1-norm condition number of applying L to the vector $x = (-2, 604)^T$.

59. Let E be the plane in R^3 defined by

$$E = \{z \in R^3 | u^T z = 0\}$$

where $u = (1, 2, 3)^T$.

(a) Find the orthogonal projection matrix P onto E .

(b) Find the closest point z in E to $b = (3, 2, 1)^T$. Note z is a 3-vector!

(c) Let

$$A = \begin{bmatrix} -2 & 0 \\ 1 & -3 \\ 0 & 2 \end{bmatrix}.$$

Find $x \in R^2$ minimizing $\|Ax - b\|_2$. Note x is a 2-vector!

(d) Compute the 2-norm condition number for the problem of applying P to any vector $b \in R^3$, in terms of the angle θ between b and E .

60. (a) Show that the condition number $\kappa_f \leq 10$ at $x = (1, 1)^T$, for the polar coordinate transform

$$f(x) = (r, r\theta)^T = (f_1(x), f_2(x))^T$$

where r and θ are defined so that $x = (x_1, x_2)^T = (r \cos \theta, r \sin \theta)^T$ with $r \geq 0$ and $|\theta| \leq \pi$. Is this problem well-conditioned?

(b) Consider the algorithm \tilde{f}_1 which evaluates $f_1(x) = r = \sqrt{x_1^2 + x_2^2}$ by multiplication, addition and square root (in that order). Show that \tilde{f}_1 is backward stable at x .

(c) Use (b) to show that \tilde{f}_1 is forward stable at x .

61. (a) Use Gaussian elimination to find a permutation matrix P , a unit lower triangular matrix L , and an upper triangular matrix R such that $PA = LR$ where

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 0 & 3 \\ 10^6 & 0 & -3 \end{bmatrix}.$$

(b) Compute the condition number of $f(x) = Rx$ where $x = (10^{-6}, 0, 1)^T$. Is f well-conditioned at x ?

62. Let E be the plane in R^3 defined by

$$E = \{z \in R^3 | v^T z = 0\}$$

where $v = (1, 2, 4)^T$.

(a) Find the matrix P which implements orthogonal projection onto E . Be careful— v is not a unit vector!

(b) Find the closest point z in E to $b = (0, 0, 1)^T$. Note z is a 3-vector!

(c) Let

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 2 \\ 0 & -1 \end{bmatrix}.$$

Find $x \in R^2$ minimizing $\|Ax - b\|_2$. Note x is a 2-vector!

(d) Compute the 2-norm condition number for the problem of multiplying $I - P$ into an arbitrary vector $b \in R^3$, in terms of the angle θ between b and E . (Hint: draw a picture.)

63. (a) Write down Newton's method for finding the root c of a single scalar equation $f(x) = 0$, and simplify it for the specific problem of finding the positive cube root c of a number $c^3 = a > 0$. (I.e. write it in an algebraic form similar to the iteration $x_{n+1} = (1/2)(x_n + a/x_n)$ for finding \sqrt{a} .)

(b) Verify that for $a = 27$ and $x_0 = 4$, the error decreases after one Newton step: $|c - x_1| < |c - x_0|$.

(c) Prove that $x_n \rightarrow c$ as $n \rightarrow \infty$ if $1 < c < x_0$. (Hint: to save algebra, do the general error estimate first, and specialize to the cube root at the last minute.)

(d) Replace $f'(x_n)$ in Newton's method by the forward difference approximation $D_h f(x) = (f(x+h) - f(x))/h$, for some small h (comparable to the error $e_n = c - x_n$ at each step). Show that the resulting iteration still converges quadratically (for a general function f with f'' bounded) if $h = O(|e_n|)$ as $n \rightarrow \infty$.

(e) Would you expect floating-point error in computer evaluation of f to ruin the conclusion of part (d)?

64. Let $P(x)$ be the quadratic polynomial that interpolates $f(x) = \cos(x)$ at $x = 9.0, 9.1$ and 9.2 (in radians).

(a) Write down the Lagrange and Newton formulas for P .

(b) Bound the relative error $e(x) = |f(x) - P(x)|/|f(x)|$ on the interval $9.0 \leq x \leq 9.2$.

(c) Bound the relative error $e(x) = |f(x) - P(x)|/|f(x)|$ on the interval $9.09 \leq x \leq 9.11$. Why is your bound so much smaller or larger than the bound in (a)?

(d) State the general formula which gives the error on an interval $[a, b]$, in the degree- n polynomial P which interpolates a smooth function f at $n + 1$ points t_j in $[a, b]$. Separate the error into three factors and explain why two of them are inevitable.

(e) When doing polynomial interpolation of smooth functions, what freedom do we have to minimize the worst-case error and how should we use it?

66. (a) Solve the 2×2 nonlinear system

$$f(x_1, x_2) = \left(\frac{1}{2}(x_1^2 - x_2^2), x_1x_2 - 1\right)^T = (0, 0)^T,$$

for $c = (x_1, x_2)$.

(b) Determine whether the fixed point iteration $x^{(k+1)} = \varphi(x^{(k)})$ with

$$\varphi(x) = \frac{1}{2}x + \frac{1}{x_1^2 + x_2^2}(x_2, x_1)^T$$

is the same as Newton's method for solving $f = 0$.

(c) Start from $x^{(0)} = (1, 0)^T$ and compute two iterates $x^{(1)}$ and $x^{(2)}$.

(d) Estimate and explain the rate of convergence observed in (c).

67. Let $P(x)$ be the quadratic polynomial that interpolates $f(x) = x^3$ at $x = 1, 2$ and 3 .

(a) Write down the Lagrange and Newton formulas for P .

(b) Bound the relative error $e(x) = |f(x) - P(x)|/|f(x)|$ on the interval $1 \leq x \leq 3$.

(c) Bound the relative error $e(x) = |f(x) - P(x)|/|f(x)|$ on the interval $1.9 \leq x \leq 2.1$. Why is your bound so much smaller or larger than the bound in (a)?

(d) State the general formula which gives the error on an interval $[a, b]$, in the degree- n polynomial P which interpolates a smooth function f at $n + 1$ points t_j in $[a, b]$. Separate the error into a product of three factors and explain why two of them are inevitable.

68. Let $B_j^n(t) = \frac{n!}{j!(n-j)!}(1-t)^{n-j}t^j$ be the j th Bernstein polynomial of degree n .

(a) Prove that $0 \leq B_j^n(t) \leq 1$ for $0 \leq t \leq 1$ and explain the usefulness of this fact.

(b) Given 4 numbers f_0, f'_0, f_1, f'_1 , find a linear combination

$$P(t) = \sum_{j=0}^3 b_j B_j^3(t)$$

of the B_j^3 's which satisfies

$$P(0) = f_0, \quad P'(0) = f'_0, \quad P(1) = f_1, \quad P'(1) = f'_1.$$

69.

Consider approximating $\cos(x)$ on the interval $[-1, 1]$. Bound the maximum error in

(a) Taylor expansion of degree 3,

(b) quadratic interpolation from x_{j-1}, x_j and x_{j+1} where $x_j = -1 + 2j/n$ is the nearest point to x from the equidistant grid of $n + 1$ points on $[-1, 1]$ with spacing $h = 2/n$, and

(c) degree- n interpolation at Chebyshev points on $[-1, 1]$.

70.

Consider the equation $x - 1 - 1/x = 0$.

(a) What are its solutions?

(b) For which solutions will the fixed point iteration

$$x_{n+1} = 1 + 1/x_n$$

converge if started close enough? For which will Newton's method converge if started close enough?

(c) In each case where convergence occurs, will it be linear or quadratic?

71.

(a) Suppose c_i and x_i are chosen so that $0 \leq x_i \leq h$ and

$$\int_0^h p(x) dx = \sum_{i=0}^n c_i p(x_i)$$

whenever p is a polynomial of degree $\leq n$. Prove that

$$\int_0^h f(x) dx = \sum_{i=0}^n c_i f(x_i) + O(h^{n+2})$$

whenever f has $n + 1$ bounded continuous derivatives.

(b) Find a numerical integration formula

$$\int_{-1}^1 f(x) dx = c_1 f(x_1) + c_2 f(x_2)$$

which is exact for $1, x, x^2$ and x^3 . Use this formula to find a fifth-order formula on the interval $[0, h]$.

72.

Consider solving the equation $f(x) = 0$ by the iteration

$$x_{n+1} = x_n - u(x_n)/u'(x_n)$$

where $u(x) = f(x)/f'(x)$. Determine the order of convergence at a root r of multiplicity 2, where

$$f(r) = f'(r) = 0 \neq f''(r).$$

73.

Find a numerical integration formula of the form

$$\int_0^h f(x)dx = Af(0) + Bf(h) + C(f'(0) - f'(h)) + O(h^q)$$

which has as high order q as possible. What is q ?

74. Let

$$m = (1/n) \sum_{i=1}^n x_i.$$

Decide which formula for the variance is likely to give a better relative error bound in floating-point arithmetic and explain the analysis behind your choice: (a)

$$(n-1)S^2 = \sum_{i=1}^n x_i^2 - nm^2$$

or (b)

$$(n-1)S^2 = \sum_{i=1}^n (x_i - m)^2.$$

75. Suppose A is an n by n real invertible matrix and $r = b - Ay$ where $b = Ax$ is nonzero. Prove that

$$\|x - y\|/\|x\| \leq \text{cond}(A)\|r\|/\|b\|.$$

76. Prove that floating point arithmetic with machine epsilon ϵ produces an inner product (summed in the natural order) satisfying

$$\text{fl}(x^T y) = x^T(y + e)$$

where

$$|e_i| \leq 2n\epsilon|y_i|,$$

as long as $n\epsilon \leq 1/2$. Under what conditions on x and y does this bound guarantee small relative error in $x^T y$?

77. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}.$$

- (a) Find a symmetric tridiagonal matrix T with the same eigenvalues as A .
- (b) Find an orthogonal matrix Q and an upper triangular matrix R with positive diagonal entries such that $T - T_{33}I = QR$.
- (c) Compute $RQ + T_{33}I$ and verify that it is closer to diagonal than T by computing the sum of squares of off-diagonal elements for both.

78. Let A be a number in the range $0.25 \leq A \leq 1$ and consider the Newton iteration

$$x_{k+1} = (x_k + A/x_k)/2$$

for computing $x_\infty = \sqrt{A}$. Assume that the initial guess $x_0 = (1 + 2A)/3$ has error less than 0.05 for any such A .

- (a) Prove that the error $e_k = x_k - \sqrt{A}$ satisfies

$$e_{k+1} = e_k^2/2x_k.$$

- (b) Determine the number k of steps necessary to guarantee sixteen-digit accuracy in \sqrt{A} .

79. (a) Let A be an unknown matrix. Given a known matrix B and known vectors p and $q = Ap$, find a rank-one update $C = B + uv^T$ of B which makes

$$\|C - A\|_2 \leq \|B - A\|_2$$

and

$$Cp = q.$$

Show that your update satisfies these two requirements.

- (b) Suppose the QR factorization of a nonsingular matrix $A = QR$ is given, where Q is orthogonal and R is upper triangular with positive diagonal elements. Under what conditions on A , u and v will the rank-one update $B = A + uv^T$ have a QR factorization and what algorithmic sequence of steps can be used to find it?

80. For an arbitrary differentiable function $F : R^n \rightarrow R^n$, write Newton's method as a fixed point iteration

$$x_{k+1} = G(x_k).$$

Determine G in terms of F and DF . Now restrict your analysis to the one-dimensional case $n = 1$, assume F has two continuous derivatives, and use Taylor expansion to prove superlinear convergence: as any two distinct points u and v approach a zero of $F(x)$ where $F'(x)$ is nonzero,

$$|G(u) - G(v)|/|u - v| \rightarrow 0.$$

81. Suppose a square matrix A has a factorization $A = U\Sigma V^T$ where U and V are orthogonal and Σ is diagonal with diagonal elements $\sigma_1 \geq \sigma_2 \geq \dots \sigma_r > 0 = 0 = \dots = 0$. Use this factorization to find a vector x which minimizes $\|x\|_2^2$ subject to the requirement that $\|Ax - b\|_2^2$ be minimum.

82. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}.$$

- (a) Find a symmetric tridiagonal matrix T with the same eigenvalues as A .
- (b) Find an orthogonal matrix Q and an upper triangular matrix R with positive diagonal entries such that $T - T_{33}I = QR$.
- (c) Compute $RQ + T_{33}I$ and verify that it is closer to diagonal than T by computing the sum of squares of off-diagonal elements for both.

83. Let A be a number in the range $0.25 \leq A \leq 1$ and consider the Newton iteration

$$x_{k+1} = (x_k + A/x_k)/2$$

for computing $x_\infty = \sqrt{A}$. Assume that the initial guess $x_0 = (1 + 2A)/3$ has error less than 0.05 for any such A .

- (a) Prove that the error $e_k = x_k - \sqrt{A}$ satisfies

$$e_{k+1} = e_k^2/2x_k.$$

- (b) Determine the number k of steps necessary to guarantee sixteen-digit accuracy in \sqrt{A} .

84. Let $a = (a_1, a_2, a_3)^T$ and $e = (1, 1, 1)^T$. For $p = 1, 2, \infty$ determine the real number x which minimizes $\|xe - a\|_p$.

85. Let A be an unknown matrix. Given a known matrix B and known vectors p and $q = Ap$, find a rank-one update $C = B + uv^T$ of B which makes

$$\|C - A\|_2 \leq \|B - A\|_2$$

and

$$Cp = q.$$

Explain why your update satisfies these two requirements.

86. Suppose the QR factorization of a nonsingular matrix $A = QR$ is given, where Q is orthogonal and R is upper triangular with positive diagonal elements. Under what conditions on A , u and v will the rank-one update $B = A + uv^T$ have a QR factorization and what algorithmic sequence of steps can be used to find it?

87. For an arbitrary differentiable function $F : R^n \rightarrow R^n$, write Newton's method as a fixed point iteration

$$x_{k+1} = G(x_k).$$

Determine G in terms of F and DF . Now restrict your analysis to the one-dimensional case $n = 1$, assume F has two continuous derivatives, and use Taylor expansion to prove superlinear convergence: as any two distinct points u and v approach a zero of $F(x)$ where $F'(x)$ is nonzero,

$$|G(u) - G(v)|/|u - v| \rightarrow 0.$$

88. Let A be an $n \times n$ matrix, $Ax = \lambda x$ and $\|x\|_\infty = 1$. Prove that λ is in one of the disks

$$R_i = \left\{ |z - a_{ii}| \leq \sum_{j \neq i} |a_{ij}| \right\}.$$

89. Suppose a square matrix A has a factorization $A = U\Sigma V^T$ where U and V are orthogonal and Σ is diagonal with diagonal elements $\sigma_1 \geq \sigma_2 \geq \dots \sigma_r > 0 = 0 = \dots = 0$. Use this factorization to find a vector x which minimizes $\|x\|_2^2$ and $\|Ax - b\|_2^2$.

89. Consider the weighted integral, $\int_{-\infty}^{\infty} \omega(x)f(x)dx$, where the weighting function is $\omega(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$. Note that if we define $I_n = \int_{-\infty}^{\infty} \omega(x)x^n dx$, then $I_0 = 1$, $I_1 = 0$, $I_2 = 1$, $I_3 = 0$, and $I_4 = 3$.

a) Design the best quadrature possible that uses evaluation points, $x_0 = 0$ and $x_1 = 1$.

d) What degree of polynomials does this integrate exactly?

a) Design a two-point *gaussian* quadrature to evaluate this integral.

b) What degree of polynomials does this integrate exactly?

90. Use an interpolating polynomial to approximate $f(x) = \cos(x)$.

- a) Write out the Lagrange basis functions you would use for continuous piecewise-linear interpolation on the interval, $[0, \frac{\pi}{2}]$. Separate the interval into two equal subintervals.
- b) What is the interpolation polynomial approximation of $\cos(x)$
- c) Construct a single cubic interpolating polynomial that matches function values and first derivatives at $x = 0$ and $x = \frac{\pi}{2}$.
91. The Cavalieri-Simpson rule for integration is $\int_a^b f(x)dx \approx (b-a)(\frac{1}{6}f(a) + \frac{2}{3}f(\frac{a+b}{2}) + \frac{1}{6}f(b))$.
- a) This quadrature correctly integrates all third degree polynomials, what is the power of H , where $H = b - a$, in the first non-zero error term?
- b) Consider a composite integration rule, $S(h)$. That is, if $h = \frac{H}{n}$, we use the Cavalieri-Simpson rule on n equal-length subintervals. What is the power of h in the first non-zero error term for $S(h)$?
- c) Use Richardson Extrapolation to eliminate the highest order error term by using step sizes: $h_1 = H$ and $h_2 = \frac{H}{2}$. Express your answer as a linear combination of $S(h_1)$ and $S(h_2)$.