**References**

1. H. Anand, V. C. Dumir, H. Gupta, A combinatorial distribution problem, Duke Math. J., 33 (1966) 757–769.
2. E. Ehrhart, Sur les carrés magiques, C. R. Acad. Sci. Paris Ser. A, 277 (1973) 651–654.
3. R. Stanley, Linear homogenous diophantine equations and magic labelings of graphs, Duke Math. J., 40 (1974) 607–632.

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## POINTWISE CONVERGENCE OF FOURIER SERIES

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The purpose of this note is to show how the usual proofs of convergence of Fourier series [1], [2] may be considerably shortened and the underlying mechanism clarified. In fact, one can get stronger conclusions with less effort.

It is best to work with complex exponentials rather than sines and cosines. Then, for a Lebesgue integrable $2\pi$-periodic function $f$, the $n$th Fourier coefficient is defined by

$$\hat{f}(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx.$$

The (unsymmetric) partial sums of the Fourier series are defined by

$$S_{m,n}(x)=\sum_{k=-m}^{n}\hat{f}(k)e^{ikx}.$$

Usually one considers only the symmetric partial sums $S_n(x)=S_{n,n}(x)$.

Just as with the standard proofs, everything turns upon one essential fact: the Riemann-Lebesgue lemma. This states that $\hat{f}(n)$ tends to 0 as $|n|$ becomes infinite. For $f$ square-integrable (e.g., piecewise continuous), Riemann-Lebesgue is an immediate consequence of the very easy Bessel inequality. The case of merely integrable $f$ follows from the square-integrable case by an approximation argument (cf. [2, Chapter 1]).

THEOREM. *Let $f$ be integrable and suppose that $f$ is differentiable at the point $x_0$. Then the Fourier partial sums $S_{m,n}(x_0)$ converge to $f(x_0)$ as $m,n\to\infty$.*

*Proof.* As usual we may suppose that $x_0=0$ and $f(x_0)=0$: just subtract a constant from $f$ and shift the origin. (The following argument would work without this preliminary reduction, of course, but the formulas would be a bit more complicated.) Since $f(0)=0$ and $f'(0)$ exists, the function $g(x)=f(x)/[e^{ix}-1]$ is bounded near 0 and thus is integrable because $f$ is integrable. Now we have

$$f(x)=(e^{ix}-1)g(x)$$

so that the Fourier coefficients satisfy

$$\hat{f}(k)=\hat{g}(k-1)-\hat{g}(k).$$

The Fourier series is a telescoping series! Indeed,

$$S_{m,n}(0)=\sum_{k=-m}^{n}\hat{f}(k)=\hat{g}(-m-1)-\hat{g}(n),$$

and this tends to 0 ($=f(0)$) by Riemann-Lebesgue. ∎

REMARKS: (1) If we assume that $f$ is piecewise continuous (the usual classroom hypothesis), then $g$ is also piecewise continuous. So one need not venture outside the piecewise continuous realm.

Similarly, if $f$ is square-integrable, so is $g$. So we need only the simplest version of the Riemann-Lebesgue lemma if we are content to work with a narrower class of functions $f$.

(2) The differentiability hypothesis is much stronger than necessary. All that is really needed for our argument to work is that $(f(x)-f(x_0))/(x-x_0)$ be Lebesgue integrable in a neighborhood of $x_0$. This is certainly the case if $f$ satisfies a Lipschitz or Hölder condition at $x_0$, e.g., if $f$ has one-sided derivatives at $x_0$.

(3) We showed that the *unsymmetric* partial sums of the Fourier series converge to $f(x_0)$. This means that the positive and negative halves of the series converge separately, a conclusion that does not follow from the usual proof.

(4) We can also deal with a jump discontinuity. Thus, suppose that $f$ has left-hand and right-hand limits at 0, which we denote $f(0^-)$ and $f(0^+)$. Suppose also that $f$ has one-sided slopes at 0, i.e., $(f(h)-f(0^+))/h$ and $(f(-h)-f(0^-))/h$ converge to limits as $h$ decreases to 0. Then the *symmetric* partial sum $S_n(0)$ converges to $[f(0^+)+f(0^-)]/2$.

*Proof.* Subtract a constant so that $f(0^+)=-f(0^-)$. We then must show that $S_n(0)$ converges to 0. Now

$$S_n(0)=\int_{-\pi}^{\pi} f(x)D_n(x)\,dx.$$

$D_n(x)=(1/2\pi)\Sigma_{-n}^{n}e^{ikx}$ is the famous "Dirichlet kernel," but all we need observe is that it is an even function. Hence

$$S_n(0)=\int_{-\pi}^{\pi} \tfrac{1}{2}[f(x)+f(-x)]D_n(x)\,dx.$$

Now we simply apply the previous theorem to the function $\frac{1}{2}[f(x)+f(-x)]$.

(5) Again the differentiability hypothesis can be weakened considerably. To deduce that $S_n(0)\to 0$ we need only the integrability of the function $[f(x)+f(-x)]/x$. It is not even necessary to assume that $f(0^+)$ and $f(0^-)$ exist.

Incidentally, the restriction to symmetric partial sums in (4) is really necessary, as one can see by considering the simplest example, that is, $f(x)=+1$ for $x>0$, $-1$ for $x<0$.

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### References

1. R. Courant and F. John, Introduction to Calculus and Analysis, vol. 1, Wiley-Interscience, New York, 1965, Section 8.4.

2. A. Zygmund, Trigonometric Series, vol. 1, 2nd ed., Cambridge, England, 1959, Chapters 1 and 2.

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## MATHEMATICAL EDUCATION

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