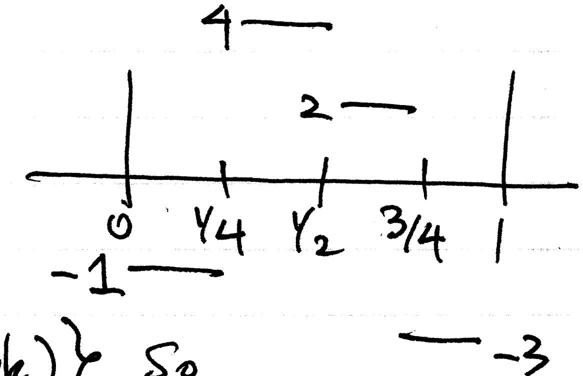
Math 118 Problem Set 10 ①

4.1, 3, 4, 5, 6, 11

$\boxed{4.1}$ $f(x) =$



$V_2 = \text{span}\{\varphi(4x-k)\}$ so

$$f(x) = (-1)\varphi(4x) + 4\varphi(4x-1) + 2\varphi(4x-2) - 3\varphi(4x-3)$$

in terms of the basis $\{\varphi(4x-k)\}$ for V_2.

Since

$$\varphi(4x) = \tfrac{1}{2}\left[\psi(2x) + \varphi(2x)\right]$$

$$\varphi(4x-1) = \varphi(4(x-\tfrac{1}{4}))$$

$$= \tfrac{1}{2}\left[\psi(2x-\tfrac{1}{2}) + \varphi(2x-\tfrac{1}{2})\right]$$

expresses $\varphi(4x-1)$ in <u>half</u> steps on the coarser level, we have to use

$$\varphi(4x-1) = \tfrac{1}{2}\left(\varphi(2x) - \psi(2x)\right)$$

instead (for odd-numbered shifts).

Hence

$$f(x) = (-1)\,\tfrac{1}{2}\left[\psi(2x) + \varphi(2x)\right]$$

$$+\, 4\,\tfrac{1}{2}\left[\varphi(2x) - \psi(2x)\right]$$

$$+\, 2\,\tfrac{1}{2}\left[\psi(2x-1) + \varphi(2x-1)\right]$$

$$+\, (-3)\,\tfrac{1}{2}\left[\varphi(2x-1) - \psi(2x-1)\right]$$

$$= +\tfrac{3}{2}\varphi(2x) - \tfrac{1}{2}\varphi(2x-1) \qquad \in V_1$$

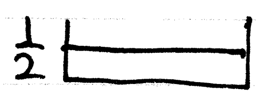
$$+\left(-\tfrac{5}{2}\right)\psi(2x) + \tfrac{5}{2}\psi(2x-1) \qquad \in W_1$$

$$= \tfrac{3}{2}\cdot\tfrac{1}{2}\left[\psi(x) + \varphi(x)\right]$$

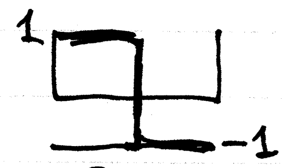
$$-\, \tfrac{1}{2}\cdot\tfrac{1}{2}\left[\varphi(x) - \psi(x)\right]$$

$$+\left(-\tfrac{5}{2}\right)\psi(2x) + \tfrac{5}{2}\psi(2x-1)$$

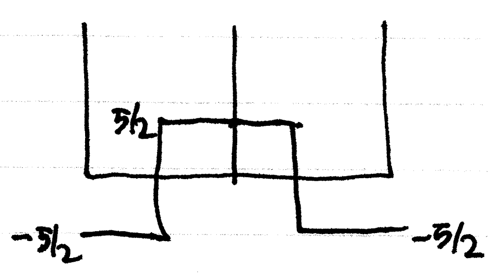
$$= \tfrac{1}{2}\varphi(x) \qquad \in V_0$$



$$+\, 1\,\psi(x) \qquad \in W_0$$



$$-\, \tfrac{5}{2}\psi(2x) + \tfrac{5}{2}\psi(2x-1) \qquad \in W_1$$



$\boxed{4.3}$ Take ON bases for A and B. Any element x of $A \oplus B$ has a unique decomposition

$$x = a + b, \quad a \in A,\ b \in B,\ a \perp b.$$

Since $a \in A$ it can be written in the ON basis for A with dim A degrees of freedom. Similarly b, so x is parametrized by dim A + dim B degrees of freedom. In other words the union of ON bases for A and B is an ON basis for $A \oplus B$. Hence $\dim(A \oplus B) = \dim A + \dim B$.

If A and B are <u>not</u> orthogonal then the union of bases spans A+B but linear independence may fail so

$$\dim(A+B) \leq \dim A + \dim B.$$

$\boxed{4.4}$ (a) Since V_0 is spanned by one element $\varphi(x)$ on $[0,1]$, it has $\dim(V_0) = 1$. Similarly

$$\dim(V_1) = \dim \operatorname{span}\{\varphi(2x), \varphi(2x-1)\}$$

$$= 2$$

$$= \dim(V_0 \oplus W_0)$$

$$= 1 + \dim(W_0).$$

Since $\dim(V_n) = 2\dim(V_{n-1})$
and $\dim(W_n) = 2\dim(W_{n-1})$
by scaling, we have

$$\dim(V_n) = \dim(W_n) = 2^n \quad n \geq 0.$$

(b)

$$\dim V_n = 2^n = 2^{n-1} + 2^{n-2} + \cdots + 2^0 + 2^0 = 2^n.$$

Yes.

$\boxed{4.5}$ $f(x) = \sum_k a_k \varphi(2x-k) \in V_1$

If $\int_{-\infty}^{\infty} f(x)\varphi(x-l) = 0 \quad \forall l \in \mathbb{Z}$ then

$$0 = \int_l^{l+1} \sum_k a_k \varphi(2x-k)\,dx$$

$$= \int_l^{l+1/2} \underset{k=2l}{a_k \varphi(2x-k)} + \int_{l+1/2}^{l+1} \underset{k=2l+1}{a_k \varphi(2x-k)}$$

$$= \tfrac{1}{2}\left(a_{2l} + a_{2l+1}\right)$$

so $\boxed{-a_{2l} = a_{2l+1}}$ for all $l \in \mathbb{Z}$.

Hence

$$f(x) = \sum_l a_{2l}\varphi(2x-2l) + \sum_l a_{2l+1}\varphi(2x-2l-1)$$

$$= \sum_l a_{2l}\left(\varphi(2x-2l) - \varphi(2x-2l-1)\right)$$

$$= \sum_l a_{2l}\psi(x-l). \in W_0$$

$k=0$

$\boxed{4.6}$ $a^2 = [1/2, 2, 5/2, -3/2]$

$b^2 = [-3/2, -1, 1/2, -1/2]$

$g \in V_3$ is given by $\quad a^3$

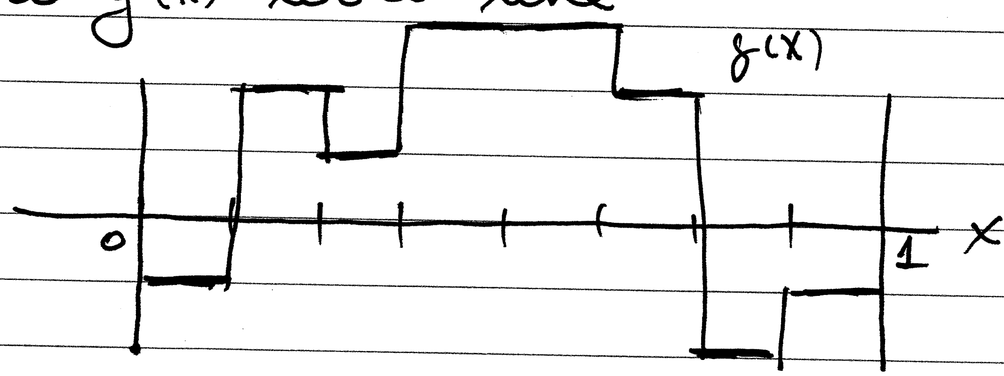
$$g(x) = \sum_k \varphi(8x-k) \qquad 8 = 2^3,$$

where

$$a_k^3 = \begin{cases} a^2_{k/2} + b^2_{k/2} & k \text{ even} \\ a^2_{(k-1)/2} - b^2_{(k-1)/2} & k \text{ odd} \end{cases}$$

$$= [-1 \quad 2 \quad 1 \quad 3 \quad 3 \quad 2 \quad -2 \quad -1]$$

So $g(x)$ looks like



$\boxed{4.11}$ $$g(x) = \begin{cases} 0 & 0 \le t < 7/17 \\ 1-t^2 & 7/17 \le t < 1 \end{cases}$$

Discretize over 64 intervals

$$g_k = 2^7 \int_{2^{-7}k}^{2^{-7}(k+1)} g(x)\,dx$$

$$= \begin{cases} 0 & 2^{-7}(k+1) < 7/17 \\ 1 \mp 2^{-14}(k^2+k+\frac{1}{3}) & 7/17 < 2^{-7}k \end{cases}$$

where $2^7 \cdot 7/17 = 52.706$

We simplify by approximating:

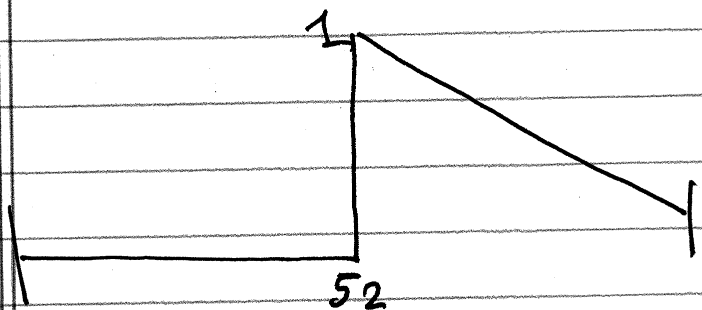
$$g_k = \begin{cases} 0 & k \le 52 \\ 1 \mp 2^{-14}k^2 & k > 52. \end{cases}$$

For a 1-level decomposition, we compute

$$b_k = g_k - g_{k-1} = \begin{cases} O(1) & k = 52 \\ -2^{-14} \cdot 2k & k > 52 \\ 0 & k < 52 \end{cases}$$

and observe that b_k is maximum at about $k=52$, where the discontinuity is.

Wavelet coefficients look like



$t = 2^{-6} \cdot 52 = 0.8125/2 = 0.40625$

close to the discontinuity at $0.411 = 7/17$.

Repeating with different t's gives similar results, but the jump is smaller when t is larger.

The method works because the wavelet coefficients localize the jump.