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(1a) Give a counterexample or sketch a proof: If $f : [-\pi, \pi] \to C$ is 2π-periodic and continuous then the Fourier series coefficients

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx$$

satisfy

$$\sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2 \le \int_{-\pi}^{\pi} |f(x)|^2\, dx.$$

<u>Proof</u>: Let $S_N f = \sum_{-N}^{N} \hat{f}(k) e^{ikx}$ be the projection of f, so $S_N^* = S_N = S_N^2$. Then

$$\|S_N f\|^2 = \langle S_N f, S_N f\rangle = \langle S_N^* S_N f, f\rangle$$
$$= \langle S_N f, f\rangle \le \|f\| \|S_N f\| \Rightarrow |S_N| \le 1$$

by Cauchy-Schwarz. Hence

$$\sum_{-N}^{N} |\hat{f}(k)|^2 = \frac{1}{2\pi} \sum_{-N}^{N}\sum_{-N}^{N} \hat{f}(k)\hat{f}(\ell) \int_{-\pi}^{\pi} e^{i(k-\ell)x}\, dx$$
$$= \|S_N f\|^2/2\pi \le \frac{1}{2\pi} \|f\|^2 \le \|f\|^2.$$

Since the terms in the left-hand sum are nonnegative, we can let $N \to \infty$ and get

$$\sum_{-\infty}^{\infty} |\hat{f}(k)|^2 \le \|f\|^2.$$

(1b) Give a counterexample or sketch a proof: If all three of $f : [-\pi, \pi] \to C$, the derivative f' and the second derivative f'' are 2π-periodic and continuous then the partial sums

$$S_N f(x) = \sum_{k=-N}^{N} \hat{f}(k) e^{ikx}$$

with coefficients

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx$$

converge pointwise to $f(x)$ for every x.

Proof: By a theorem in the text, the Fourier series converges to $f(x)$ at every x where $f'(x)$ exists and f is continuous. Sketch: Integration by parts bounds the tail:

$$|\hat{f}(k)| \le \frac{C}{k^2} \Rightarrow |(S_\infty - S_N)f| \le \frac{C}{N}$$

but we need to know that the limit is f, and the Dirichlet kernel formula is the way to do it:

$$f(x) - S_N f(x) = \int_{-\pi}^{\pi} \frac{\sin((N+1/2)(x-y))}{\sin(x-y)/2} (f(x) - f(y))\, dy.$$

Write

$$|f(x) - f(y)| = \left|\int_x^y f'(z)\, dz\right| \le C|x-y|$$

and change variables. Use dominated convergence to show $f - S_N f \to 0$ pointwise.

(1c) Give a counterexample or sketch a proof: If $f : [-\pi, \pi] \to C$ is 2π-periodic and

$$\int_{-\pi}^{\pi} |f(x)|^2 \, dx < \infty$$

then the partial sums

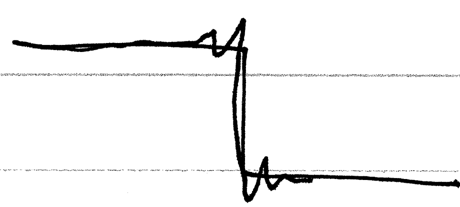
$$S_N f(x) = \sum_{k=-N}^{N} \hat{f}(k) e^{ikx}$$

with coefficients

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} \, dx$$

converge uniformly to $f(x)$ on the interval $[-\pi, \pi]$.

<u>Counterexample</u>: Take any discontinuous L^2 function such as $f(x) = \text{sgn}(x)$. Its Fourier series has continuous partial sums which cannot converge uniformly to a descontinuous function. They avoid uniform convergence by the Gibbs phenomenon, giving a constant overshoot/undershoot near jumps in a smaller and smaller region around the jump.



(2) Suppose real-valued continuous functions e_0, e_1, e_2, ... are an orthonormal set in $L^2[0,1]$ and $e_0(x) > 0$ for all x. Show that there must be a point $x = x_{11}$ in $[0,1]$ where $e_1(x) = 0$.

Otherwise WLOG $e_1(x) > 0$ for every x so $\int_0^1 e_0(x)e_1(x)\,dx > 0$ since e_0 and e_1 are continuous. This would contradict orthonormality.

Note that without the hypothesis of real values e^{ikx} provides a counterexample, and without continuity ___ provides a counterexample.

(3a) Compute the complex Fourier coefficients of

$$f(x) = e^{-|x|}$$

on the interval $-\pi < x < \pi$.

$$\frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-|x|} e^{-ikx}\, dx = \hat{f}(k) =$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{-\pi}^{0} e^{x} e^{-ikx}\, dx + \int_{0}^{\pi} e^{-x} e^{-ikx}\, dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{(1-ik)x}}{1-ik} \Big|_{-\pi}^{0} + \frac{e^{-(1+ik)x}}{-(1+ik)} \Big|_{0}^{\pi} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1 - e^{-(1-ik)\pi}}{1-ik} + \frac{1 - e^{-(1+ik)\pi}}{1+ik} \right)$$

$$= \sqrt{\frac{2}{\pi}} \; \frac{1 - (-1)^k e^{-\pi}}{1+k^2}.$$

This is an even function of k since $f(x)$ is even, and real since f is also real (and even).

(3b) Compute the complex Fourier transform of

$$f(x) = e^{-|x|}$$

on the interval $-\infty < x < \infty$.

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{0} e^{x} e^{-ikx} + \int_{0}^{\infty} e^{-x} e^{-ikx}\right)$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{1}{1+k^2}$$

since the previous formula 3a applies with $e^{-\pi}$ replaced by $e^{-\infty} = 0$.

(3c) Evaluate the integrals

$$\int_0^\infty \frac{1}{1+k^2}\,dk$$

and

$$\int_0^\infty \frac{1}{(1+k^2)^2}\,dk.$$

By the Fourier inversion formula,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\frac{1}{1+k^2}\cancel{dk}\, e^{ikx}\,dk = e^{-|x|}$$

So for $x=0$ we find

$$\int_{-\infty}^{\infty}\frac{1}{1+k^2}\,dk = \pi$$

and since $1/(1+k^2)$ is even

$$\int_0^\infty \frac{1}{1+k^2}\,dk = \boxed{\pi/2}.$$

By Parseval's Theorem, $\|f\|^2 = \|\hat{f}\|^2$ so

$$\int_{-\infty}^{\infty} e^{-2|x|}\,dx = 2\int_0^\infty e^{-2x}\,dx = 1$$

and

$$1 = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{1+k^2}\right)^2 dk = \frac{4}{\pi}\int_0^\infty\left(\frac{1}{1+k^2}\right)^2 dk$$

So

$$\int_0^\infty\left(\frac{1}{1+k^2}\right)^2 dk = \boxed{\frac{\pi}{4}.}$$

(3d) Given a smooth function f on the interval $-\infty < x < \infty$, find an integral formula $u = K * f$ for a smooth solution u of the differential equation

$$-u'' + u = f.$$

Fourier transform gives

$$(1+k^2)\hat{u}(k) = \hat{f}(k)$$

or $$u(x) = \int_{-\infty}^{\infty} K(x-y)\, f(y)\, dy \,/\sqrt{2\pi}$$

where $\hat{K}(k) = 1/1+k^2$ and by 3b

$$K(x) = \sqrt{\frac{\pi}{2}}\, e^{-|x|}.$$

$$K(x)/\sqrt{2\pi} = \frac{1}{2} e^{-|x|}.$$

Hence $$\boxed{u(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|} f(y)\, dy.}$$

<u>Check</u>:

$$u(x) = \frac{1}{2}\int_{-\infty}^{x} e^{-(x-y)} f(y) + \frac{1}{2}\int_{x}^{\infty} e^{(x-y)} f(y)\, dy$$

$$= \frac{1}{2} e^{-x} \int_{-\infty}^{x} e^{y} f(y)\, dy + \frac{1}{2} e^{x} \int_{x}^{\infty} e^{-y} f(y)\, dy$$

$\vdots$

$$u''(x) = u - f(x).$$