Math 118 Spring 2003: Supplementary Exercises 1

Exercise 1 *Use the Dominated Convergence Theorem to prove that convolution with a scaled Gaussian converges to the original function as the scaling goes to zero:*

$$\frac{1}{\sqrt{\pi\epsilon}}\int_{-\infty}^{\infty} e^{-(x-y)^2/\epsilon} f(y)dy \to f(x) \qquad as \qquad \epsilon\to 0.$$

(Hint: change variables to $z = (x-y)/\sqrt{\epsilon}$*.)*

Exercise 2 *Verify that the complex 2-dimensional vector space* $\mathbf{C}^2$ *is a Hilbert space under the usual inner product*

$$< u, v >= \sum_{j=1}^{2} u_j^* v_j :$$

Check that (a) the associated norm

$$\|u\| = \sqrt{< u, u >}$$

has properties A.3 through A.5, (b) that Cauchy sequences converge, and (c) that A.6 is satisfied.

Exercise 3 *Let* $H = L^2(\mathbf{R})$ *and* $L : H\to H$ *be the integral operator*

$$Lf(x) = \int_{-\infty}^{\infty} e^{-|x-y|} f(y)dy.$$

Compute the adjoint operator L^**.*

Exercise 4 *Let* $H = L^2(\mathbf{R})$ *and* $L : H\to H$ *be the operator that takes* f *to* $f(x)$ *if* $x > 0$ *and 0 otherwise. Determine whether* L *is (a) a projector, (b) an orthogonal projector, and (c) selfadjoint.*