The triple helix

John R. Steel
University of California, Berkeley

October 2010

Three staircases

Plan:

I. The interpretability hierarchy.
II. The vision of "ultimate K".
III. The triple helix.
IV. Some locator axioms.
V. Some basic open problems.

The Interpretability Hierarchy

For T a theory, let Arithmetic_T be the set of consequences of T in the language of first order arithmetic.

The Interpretability Hierarchy

For T a theory, let Arithmetic_T be the set of consequences of T in the language of first order arithmetic.

Phenomenon: If T is a natural extension of ZFC, then there is an extension S axiomatized by large cardinal hypotheses such that

$$\text{Arithmetic}_T = \text{Arithmetic}_S.$$

The Interpretability Hierarchy

For T a theory, let Arithmetic_T be the set of consequences of T in the language of first order arithmetic.

Phenomenon: If T is a natural extension of ZFC, then there is an extension S axiomatized by large cardinal hypotheses such that

$$\text{Arithmetic}_T = \text{Arithmetic}_S.$$

Moreover, if T and U are natural extensions of ZFC, then

$$\text{Arithmetic}_T \subseteq \text{Arithmetic}_U,$$

or

$$\text{Arithmetic}_U \subseteq \text{Arithmetic}_T.$$

The Interpretability Hierarchy

For T a theory, let $\mathrm{Arithmetic}_T$ be the set of consequences of T in the language of first order arithmetic.

Phenomenon: If T is a natural extension of ZFC, then there is an extension S axiomatized by large cardinal hypotheses such that

$$\mathrm{Arithmetic}_T = \mathrm{Arithmetic}_S.$$

Moreover, if T and U are natural extensions of ZFC, then

$$\mathrm{Arithmetic}_T \subseteq \mathrm{Arithmetic}_U,$$

or

$$\mathrm{Arithmetic}_U \subseteq \mathrm{Arithmetic}_T.$$

In practice, $\mathrm{Arithmetic}_T \subseteq \mathrm{Arithmetic}_U$ iff PA proves $\mathrm{Con}(U) \Rightarrow \mathrm{Con}(T)$.

Definition

For any theory T in LST, 1-Arithmetic$_T$ is the set of consequences of T of the form "$V_{\omega+1} \models \varphi$".

Definition

For any theory T in LST, 1-Arithmetic$_T$ is the set of consequences of T of the form "$V_{\omega+1} \models \varphi$".

Let S be the theory ZFC plus "there are infinitely many Woodin cardinals."

Definition

For any theory T in LST, 1-Arithmetic$_T$ is the set of consequences of T of the form "$V_{\omega+1} \models \varphi$".

Let S be the theory ZFC plus "there are infinitely many Woodin cardinals."

Phenomenon: Let T, U be natural theories such that Arithmetic$_S \subseteq$ Arithmetic$_T$ and Arithmetic$_S \subseteq$ Arithmetic$_U$. Then either

$$\text{1-Arithmetic}_T \subseteq \text{1-Arithmetic}_U,$$

or

$$\text{1-Arithmetic}_U \subseteq \text{1-Arithmetic}_T.$$

Definition

For any theory T in LST, 1-Arithmetic$_T$ is the set of consequences of T of the form "$V_{\omega+1} \models \varphi$".

Let S be the theory ZFC plus "there are infinitely many Woodin cardinals."

Phenomenon: Let T, U be natural theories such that Arithmetic$_S \subseteq$ Arithmetic$_T$ and Arithmetic$_S \subseteq$ Arithmetic$_U$. Then either

$$\text{1-Arithmetic}_T \subseteq \text{1-Arithmetic}_U,$$

or

$$\text{1-Arithmetic}_U \subseteq \text{1-Arithmetic}_T.$$

So at the level of sentences about $V_{\omega+1}$, we know of only **one road upward**. We are led to it many different ways. Strong axioms of infinity are its central markers.
CH is a sentence about $V_{\omega+2}$.

Remarks.

1. Directedness, not linearity, is what is important for a "well-determined" theory of $(V_{\omega+1}, \in)$.

Remarks.

1. Directedness, not linearity, is what is important for a "well-determined" theory of $(V_{\omega+1}, \in)$.
2. The phenomenon extends to sentences of the form "$\exists A \in Hom_\infty (V_{\omega+1}, \in, A) \models \psi$)." These are called $(\Sigma^2_1)^{Hom_\infty}$ *statements*.

Remarks.

1. Directedness, not linearity, is what is important for a "well-determined" theory of $(V_{\omega+1}, \in)$.
2. The phenomenon extends to sentences of the form "$\exists A \in Hom_\infty (V_{\omega+1}, \in, A) \models \psi$)." These are called $(\Sigma^2_1)^{Hom_\infty}$ *statements*.
3. By Levy-Solovay, it does not extend to arbitrary Σ^2_1 statements like CH, *if* our consistency-strength lower bound is one of the conventional large cardinals. These are preserved by small forcing, so cannot decide CH.

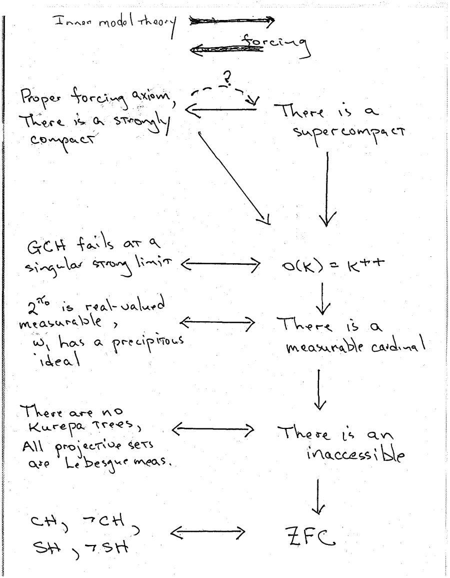
Remarks.

1. Directedness, not linearity, is what is important for a "well-determined" theory of $(V_{\omega+1}, \in)$.
2. The phenomenon extends to sentences of the form "$\exists A \in Hom_\infty(V_{\omega+1}, \in, A) \models \psi$)." These are called $(\Sigma^2_1)^{Hom_\infty}$ *statements*.
3. By Levy-Solovay, it does not extend to arbitrary Σ^2_1 statements like CH, *if* our consistency-strength lower bound is one of the conventional large cardinals. These are preserved by small forcing, so cannot decide CH.
4. We shall stick to the initial segment of the interpretability hierarchy where the natural markers are conventional large cardinals, preserved by small forcing.

Remarks.

1. Directedness, not linearity, is what is important for a "well-determined" theory of $(V_{\omega+1}, \in)$.
2. The phenomenon extends to sentences of the form "$\exists A \in Hom_\infty (V_{\omega+1}, \in, A) \models \psi$)." These are called $(\Sigma^2_1)^{Hom_\infty}$ *statements*.
3. By Levy-Solovay, it does not extend to arbitrary Σ^2_1 statements like CH, *if* our consistency-strength lower bound is one of the conventional large cardinals. These are preserved by small forcing, so cannot decide CH.
4. We shall stick to the initial segment of the interpretability hierarchy where the natural markers are conventional large cardinals, preserved by small forcing.

Some equiconsistencies



Further up

33

Strong compact, forcing axioms PFA, MM $\longleftarrow$ supercompact

$\downarrow$

$\neg\square_\kappa$, for κ singular strong limit $\longleftarrow$ many superstrongs

ω_1-dense ideal

ZF+AD $\longleftrightarrow$ ω Woodins

ω_2 saturated ial on ω_1

Δ^1_2-det $\longleftrightarrow$ one Woodin cardinal

The vision of ultimate K.

What does this picture say about what we should believe, or give preferred development, as a framework theory?

The vision of ultimate K.

What does this picture say about what we should believe, or give preferred development, as a framework theory?

1. Developing one theory develops them all, via the relative interpretations. At the level of concrete statements, they all agree.

The vision of ultimate K.

What does this picture say about what we should believe, or give preferred development, as a framework theory?

1. Developing one theory develops them all, via the relative interpretations. At the level of concrete statements, they all agree.
2. But there might be a best framework theory, or equivalence class of them under inter-translations.

The vision of ultimate K.

What does this picture say about what we should believe, or give preferred development, as a framework theory?

1. Developing one theory develops them all, via the relative interpretations. At the level of concrete statements, they all agree.
2. But there might be a best framework theory, or equivalence class of them under inter-translations.
3. It might contain a *locator axiom* (saying which staircase you are on), and *height axioms* (saying how far up you are). Its incompleteness might reside solely in the height axioms. So it would decide CH, and much more.

The vision of ultimate K.

What does this picture say about what we should believe, or give preferred development, as a framework theory?

1. Developing one theory develops them all, via the relative interpretations. At the level of concrete statements, they all agree.
2. But there might be a best framework theory, or equivalence class of them under inter-translations.
3. It might contain a *locator axiom* (saying which staircase you are on), and *height axioms* (saying how far up you are). Its incompleteness might reside solely in the height axioms. So it would decide CH, and much more.
4. The best location is the center! It is easier to leave a canonical inner model by forcing than to get back into one by core model theory.

(5) Nothing rivals fine structure theory as a global theory. There is no candidate locator axiom which produces a detailed, global theory of the universe of sets, except those asserting that V is some kind of canonical inner model.

(5) Nothing rivals fine structure theory as a global theory. There is no candidate locator axiom which produces a detailed, global theory of the universe of sets, except those asserting that V is some kind of canonical inner model.

(6) If you can make precise sense of "$V =$ ultimate K", and it is compatible with all consistency strengths up to rank-to-rank embeddings, then it has many virtues as a locator:

(5) Nothing rivals fine structure theory as a global theory. There is no candidate locator axiom which produces a detailed, global theory of the universe of sets, except those asserting that V is some kind of canonical inner model.

(6) If you can make precise sense of "$V =$ ultimate K", and it is compatible with all consistency strengths up to rank-to-rank embeddings, then it has many virtues as a locator:

    (a) everything there is is arranged in a finely-grained, harmoniously ordered hierarchy,

(5) Nothing rivals fine structure theory as a global theory. There is no candidate locator axiom which produces a detailed, global theory of the universe of sets, except those asserting that V is some kind of canonical inner model.

(6) If you can make precise sense of "$V =$ ultimate K", and it is compatible with all consistency strengths up to rank-to-rank embeddings, then it has many virtues as a locator:

    (a) everything there is is arranged in a finely-grained, harmoniously ordered hierarchy,

    (b) nothing appears until you are fully ready to understand it,

(5) Nothing rivals fine structure theory as a global theory. There is no candidate locator axiom which produces a detailed, global theory of the universe of sets, except those asserting that V is some kind of canonical inner model.

(6) If you can make precise sense of "$V =$ ultimate K", and it is compatible with all consistency strengths up to rank-to-rank embeddings, then it has many virtues as a locator:

    (a) everything there is is arranged in a finely-grained, harmoniously ordered hierarchy,
    (b) nothing appears until you are fully ready to understand it,
    (c) when you are ready, it does appear! (Unlike L.)

(5) Nothing rivals fine structure theory as a global theory. There is no candidate locator axiom which produces a detailed, global theory of the universe of sets, except those asserting that V is some kind of canonical inner model.

(6) If you can make precise sense of "$V =$ ultimate K", and it is compatible with all consistency strengths up to rank-to-rank embeddings, then it has many virtues as a locator:

   (a) everything there is is arranged in a finely-grained, harmoniously ordered hierarchy,

   (b) nothing appears until you are fully ready to understand it,

   (c) when you are ready, it does appear! (Unlike L.)

   (d) In particular, you see natural models for all other natural theories. For example, you will find plenty of ways to enter PFA- worlds with the same theory of the concrete as your own.

(5) Nothing rivals fine structure theory as a global theory. There is no candidate locator axiom which produces a detailed, global theory of the universe of sets, except those asserting that V is some kind of canonical inner model.

(6) If you can make precise sense of "$V =$ ultimate K", and it is compatible with all consistency strengths up to rank-to-rank embeddings, then it has many virtues as a locator:
   (a) everything there is is arranged in a finely-grained, harmoniously ordered hierarchy,
   (b) nothing appears until you are fully ready to understand it,
   (c) when you are ready, it does appear! (Unlike L.)
   (d) In particular, you see natural models for all other natural theories. For example, you will find plenty of ways to enter PFA- worlds with the same theory of the concrete as your own.

(7) Parallel: if we were to show $0^\sharp$ does not exist, then $V = L$ would become a natural locator axiom.

The triple helix

The initial segment of the consistency-strength hierarchy we understand well enough to prove nontrivial equiconsistencies is $\leq$ a Woodin limit of Woodin cardinals. (Yes, that's all!) In this region:

The triple helix

The initial segment of the consistency-strength hierarchy we understand well enough to prove nontrivial equiconsistencies is $\leq$ a Woodin limit of Woodin cardinals. (Yes, that's all!) In this region:

1. At the center are 3 intertwined hierarchies of sets. Staircase 1 is the Wadge hierarchy in a canonical model of AD containing all reals. Staircases 2 and 3 are the fine-structural hierarchies of two inter-related types of canonical inner model for ZFC.

The triple helix

The initial segment of the consistency-strength hierarchy we understand well enough to prove nontrivial equiconsistencies is $\leq$ a Woodin limit of Woodin cardinals. (Yes, that's all!) In this region:

1. At the center are 3 intertwined hierarchies of sets. Staircase 1 is the Wadge hierarchy in a canonical model of AD containing all reals. Staircases 2 and 3 are the fine-structural hierarchies of two inter-related types of canonical inner model for ZFC.
2. Models in each hierarchy see the others at levels $\leq$ their own.

The triple helix

The initial segment of the consistency-strength hierarchy we understand well enough to prove nontrivial equiconsistencies is $\leq$ a Woodin limit of Woodin cardinals. (Yes, that's all!) In this region:

1. At the center are 3 intertwined hierarchies of sets. Staircase 1 is the Wadge hierarchy in a canonical model of AD containing all reals. Staircases 2 and 3 are the fine-structural hierarchies of two inter-related types of canonical inner model for ZFC.
2. Models in each hierarchy see the others at levels $\leq$ their own.
3. You can't develop the theory of one hierarchy without developing the theory of all three. You can't prove consistency strength lower bounds without constructing all three types of model simultaneously.

The triple helix

The initial segment of the consistency-strength hierarchy we understand well enough to prove nontrivial equiconsistencies is $\leq$ a Woodin limit of Woodin cardinals. (Yes, that's all!) In this region:

1. At the center are 3 intertwined hierarchies of sets. Staircase 1 is the Wadge hierarchy in a canonical model of AD containing all reals. Staircases 2 and 3 are the fine-structural hierarchies of two inter-related types of canonical inner model for ZFC.
2. Models in each hierarchy see the others at levels $\leq$ their own.
3. You can't develop the theory of one hierarchy without developing the theory of all three. You can't prove consistency strength lower bounds without constructing all three types of model simultaneously.
4. One of the 3 types of models may be "preferred".

Staircase 1

Definition

A set $A \subseteq \omega^\omega$ is Hom_∞ iff for any κ, there is a continuous function π on ω^ω such that for all x, $\pi(x)$ is a tower of κ-complete measures, and

$$x \in A \Leftrightarrow \pi(x) \text{ is wellfounded.}$$

Staircase 1

Definition

A set $A \subseteq \omega^\omega$ is Hom_∞ iff for any κ, there is a continuous function π on ω^ω such that for all x, $\pi(x)$ is a tower of κ-complete measures, and

$$x \in A \Leftrightarrow \pi(x) \text{ is wellfounded.}$$

The concept comes from Martin 1968. Hom_∞ sets are determined. The definition seems to capture what it is about sets of reals that makes them "well-behaved".

Staircase 1

Definition

A set $A \subseteq \omega^\omega$ is *Hom*$_\infty$ iff for any κ, there is a continuous function π on ω^ω such that for all x, $\pi(x)$ is a tower of κ-complete measures, and

$$x \in A \Leftrightarrow \pi(x) \text{ is wellfounded.}$$

The concept comes from Martin 1968. *Hom*$_\infty$ sets are determined. The definition seems to capture what it is about sets of reals that makes them "well-behaved".

Theorem (Martin, S., Woodin)

If there are arbitrarily large Woodin cardinals, then for any pointclass Γ properly contained in Hom$_\infty$, every set of reals in $L(\Gamma, \mathbb{R})$ is in Hom$_\infty$, and thus $L(\Gamma, \mathbb{R}) \models \mathrm{AD}$.

Theorem (Woodin)

If there are arbitrarily large Woodin cardinals, then $(\Sigma^2_1)^{Hom_\infty}$ *statements are absolute for set forcing.*

Theorem (Woodin)

If there are arbitrarily large Woodin cardinals, then $(\Sigma^2_1)^{Hom_\infty}$ statements are absolute for set forcing.

In practice, generic absoluteness of a class of statements can be proved by reducing them to $(\Sigma^2_1)^{Hom_\infty}$ statements. (You may need more than arbitrarily large Woodin cardinals to do that!)

Theorem (Woodin)

If there are arbitrarily large Woodin cardinals, then $(\Sigma^2_1)^{Hom_\infty}$ statements are absolute for set forcing.

In practice, generic absoluteness of a class of statements can be proved by reducing them to $(\Sigma^2_1)^{Hom_\infty}$ statements. (You may need more than arbitrarily large Woodin cardinals to do that!)
There is an approximation to being Hom_∞ which can be used in constructing the sets in staircase 1 in universes where we have no measurable cardinals.

Staircase 2

Definition

A *pure extender model* is the constructible closure of a *coherent sequence of extenders*.

Staircase 2

Definition

A *pure extender model* is the constructible closure of a *coherent sequence of extenders*.

- An *extender* is a system of ultrafilters coding an elementary embedding.

Staircase 2

Definition

A *pure extender model* is the constructible closure of a *coherent sequence of extenders*.

- An *extender* is a system of ultrafilters coding an elementary embedding. The extenders in a coherent sequence appear in order of their strength, without leaving gaps.

Staircase 2

Definition

A *pure extender model* is the constructible closure of a *coherent sequence of extenders*.

- An *extender* is a system of ultrafilters coding an elementary embedding.The extenders in a coherent sequence appear in order of their strength, without leaving gaps.
- What makes a countable pure extender model M canonical is the existence of a Hom_∞ iteration strategy for M. This is a $(\Sigma^2_1)^{Hom_\infty}$ statement about M.

Staircase 2

Definition

A *pure extender model* is the constructible closure of a *coherent sequence of extenders*.

- An *extender* is a system of ultrafilters coding an elementary embedding.The extenders in a coherent sequence appear in order of their strength, without leaving gaps.
- What makes a countable pure extender model M canonical is the existence of a Hom_∞ iteration strategy for M. This is a $(\Sigma^2_1)^{Hom_\infty}$ statement about M.It is reasonable to hope that all $(\Sigma^2_1)^{Hom_\infty}$ statements ψ can be reduced to statements of the form " there is a Hom_∞ iteration strategy for some $M \models f(\psi)$."

Staircase 2

Definition

A *pure extender model* is the constructible closure of a *coherent sequence of extenders*.

- An *extender* is a system of ultrafilters coding an elementary embedding.The extenders in a coherent sequence appear in order of their strength, without leaving gaps.
- What makes a countable pure extender model M canonical is the existence of a Hom_∞ iteration strategy for M. This is a $(\Sigma^2_1)^{Hom_\infty}$ statement about M.It is reasonable to hope that all $(\Sigma^2_1)^{Hom_\infty}$ statements ψ can be reduced to statements of the form " there is a Hom_∞ iteration strategy for some $M \models f(\psi)$."
- Every real in an iterable extender model is ordinal definable.

Staircase 2

Definition

A *pure extender model* is the constructible closure of a *coherent sequence of extenders*.

- An *extender* is a system of ultrafilters coding an elementary embedding.The extenders in a coherent sequence appear in order of their strength, without leaving gaps.
- What makes a countable pure extender model M canonical is the existence of a Hom_∞ iteration strategy for M. This is a $(\Sigma^2_1)^{Hom_\infty}$ statement about M.It is reasonable to hope that all $(\Sigma^2_1)^{Hom_\infty}$ statements ψ can be reduced to statements of the form " there is a Hom_∞ iteration strategy for some $M \models f(\psi)$."
- Every real in an iterable extender model is ordinal definable.
- At the moment, we can only construct iteration strategies for M a bit past Woodin limits of Woodins. The fine structure theory for iterable M works up through superstrong cardinals.

Staircase 3

Leaning heaviliy on work of Woodin:

Theorem

No iterable pure extender model with a Woodin cardinal satisfies "I am iterable".

Staircase 3

Leaning heaviliy on work of Woodin:

Theorem

No iterable pure extender model with a Woodin cardinal satisfies "I am iterable".

Psuedo-definition: A *hod mouse* is a model constructed from a coherent sequence of extenders, together with an iteration strategy for the model.

Staircase 3

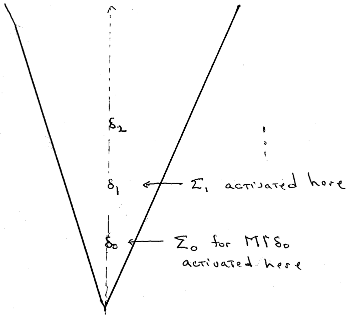
Leaning heaviliy on work of Woodin:

Theorem

No iterable pure extender model with a Woodin cardinal satisfies "I am iterable".

Psuedo-definition: A *hod mouse* is a model constructed from a coherent sequence of extenders, together with an iteration strategy for the model.

The concept was made precise by Woodin for hod mice having countably many Woodin cardinals. Grigor Sargsyan developed it further, up to measurable cardinals which are limits of Woodins. Steel and Sargsyan have gone somewhat beyond that.



A hod mouse M

$\delta_\xi = \xi^{th}$ Woodin of M

Σ_ξ = iteration strategy for $M|\delta_\xi$

Some connections

Theorem (Sargsyan 2008)

Assume AD^+ *and there is no model of* $\mathrm{AD}_{\mathbb{R}}$ + "θ *is measurable" containing all the reals; then HOD is a hod mouse.*

Some connections

Theorem (Sargsyan 2008)

Assume AD^+ *and there is no model of* $\mathsf{AD}_\mathbb{R}$ + "θ *is measurable" containing all the reals; then HOD is a hod mouse.*

For determinacy models M, HOD^M knows the theory of M. E.g.:

Theorem (Woodin late 80's)

Assume $\mathsf{AD}_\mathbb{R}$ + "θ *is regular"; then* V *is elementarily embeddable into a symmetric extension of HOD.*

Some connections

Theorem (Sargsyan 2008)

Assume AD^+ *and there is no model of* $\mathrm{AD}_{\mathbb{R}}$ + "θ *is measurable" containing all the reals; then HOD is a hod mouse.*

For determinacy models M, HOD^M knows the theory of M. E.g.:

Theorem (Woodin late 80's)

Assume $\mathrm{AD}_{\mathbb{R}}$ + "θ *is regular"; then* V *is elementarily embeddable into a symmetric extension of HOD.*

Mouse Set Conjecture: Assume AD^+ and there is no iterable pure extender model with a superstrong; then HOD is a pure extender model below its least Woodin cardinal.

Some connections

Theorem (Sargsyan 2008)

Assume AD^+ *and there is no model of* $\mathrm{AD}_{\mathbb{R}}$ + "θ *is measurable*" *containing all the reals; then HOD is a hod mouse.*

For determinacy models M, HOD^M knows the theory of M. E.g.:

Theorem (Woodin late 80's)

Assume $\mathrm{AD}_{\mathbb{R}}$ + "θ *is regular"; then* V *is elementarily embeddable into a symmetric extension of HOD.*

Mouse Set Conjecture: Assume AD^+ and there is no iterable pure extender model with a superstrong; then HOD is a pure extender model below its least Woodin cardinal.
Sargsyan proved this for determinacy models in the region to which his work applied.

The core model induction

One proves consistency strength lower bounds for theories like PFA by climbing all three staircases together. A sample theorem:

The core model induction

One proves consistency strength lower bounds for theories like PFA by climbing all three staircases together. A sample theorem:

Theorem (Woodin 90's, Sargsyan 2008)

The following are equiconsistent

(1) ZFC + *"there is an ω_1-dense ideal on ω_1* + CH + $(*)$,

(2) $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ + *"Θ is regular".*

PFA, MM, strong compact $\longleftarrow$ supercompact

$\neg\Box_\kappa$ $\longleftarrow$ many superstrongs

?

divergent models of $AD^+ + \Theta_0 < \Theta$ $\longleftrightarrow$ Woodin limit of Woodins

ω_1-dense ideal $+ CH$ $+ (*)$ $AD_{\mathbb{R}} + \Theta$ regular $\dashleftarrow\dashrightarrow$?

$AD^+ + \Theta_{\omega_1} < \Theta$ $\longleftrightarrow$ $AD^+ + \Theta_{\omega_1} < \Theta$ hypo

$AD_{\mathbb{R}} + DC$ $\longleftrightarrow$ $AD_{\mathbb{R}} + DC$ - hypo

$AD_{\mathbb{R}}$ $\longleftrightarrow$ $AD_{\mathbb{R}}$ - hypo

ω_1-dense ideal, AD $\longleftrightarrow$ ω Woodins

Some locator axioms

(*A*1) There is a strong cardinal, and arbitrarily large Woodin cardinals, and for κ the least strong cardinal, and M the *derived model* of V at κ, there is an elementary

$$j\colon V_\kappa \to \mathrm{HOD}^M|\theta.$$

Some locator axioms

$(A1)$ There is a strong cardinal, and arbitrarily large Woodin cardinals, and for κ the least strong cardinal, and M the *derived model* of V at κ, there is an elementary

$$j\colon V_\kappa \to \mathrm{HOD}^M|\theta.$$

$(A2)_n$ There are arbitrarily large Woodins, and $V_\kappa \prec_{\Sigma_n} V$, such that for M the derived model of V at κ, there is an elementary

$$j\colon V_\kappa \to \mathrm{HOD}^M|\theta.$$

Some locator axioms

(*A*1) There is a strong cardinal, and arbitrarily large Woodin cardinals, and for κ the least strong cardinal, and M the *derived model* of V at κ, there is an elementary

$$j\colon V_\kappa \to \mathrm{HOD}^M|\theta.$$

$(A2)_n$ There are arbitrarily large Woodins, and $V_\kappa \prec_{\Sigma_n} V$, such that for M the derived model of V at κ, there is an elementary

$$j\colon V_\kappa \to \mathrm{HOD}^M|\theta.$$

(A2) implies there are no strong cardinals, but may be compatible with all the local forms of large cardinal axioms. (Having a strong cardinal is like having a largest rank.)

Some locator axioms

(A1) There is a strong cardinal, and arbitrarily large Woodin cardinals, and for κ the least strong cardinal, and M the *derived model* of V at κ, there is an elementary

$$j\colon V_\kappa \to \mathrm{HOD}^M|\theta.$$

$(A2)_n$ There are arbitrarily large Woodins, and $V_\kappa \prec_{\Sigma_n} V$, such that for M the derived model of V at κ, there is an elementary

$$j\colon V_\kappa \to \mathrm{HOD}^M|\theta.$$

(A2) implies there are no strong cardinals, but may be compatible with all the local forms of large cardinal axioms. (Having a strong cardinal is like having a largest rank.)
Both (A1) and (A2) say that V looks like the HOD of an AD model.

$(A3)_n$ For arbitrarily large α, V is Σ_n equivalent to the α-complete backgrounded pure extender model.

- $(A3)_n$ For arbitrarily large α, V is Σ_n equivalent to the α-complete backgrounded pure extender model.
- $(A4)$ $\mathsf{AD}_{\mathbb{R}}$ + "θ is regular" + "there is lots of stuff above θ".

Some basic open problems

By now, philosophy has far outrun the math. Some problems:

Some basic open problems

By now, philosophy has far outrun the math. Some problems:

(1) How iterable is V? Are there iterable pure extender models having superstrongs? Measurable Woodin cardinals? Supercompacts?

Some basic open problems

By now, philosophy has far outrun the math. Some problems:

(1) How iterable is V? Are there iterable pure extender models having superstrongs? Measurable Woodin cardinals? Supercompacts?

(2) Are there hod mice with Woodin limits of Woodin cardinals?

Some basic open problems

By now, philosophy has far outrun the math. Some problems:

(1) How iterable is V? Are there iterable pure extender models having superstrongs? Measurable Woodin cardinals? Supercompacts?

(2) Are there hod mice with Woodin limits of Woodin cardinals?

(3) Is the Mouse Set Conjecture true?

Some basic open problems

By now, philosophy has far outrun the math. Some problems:

(1) How iterable is V? Are there iterable pure extender models having superstrongs? Measurable Woodin cardinals? Supercompacts?

(2) Are there hod mice with Woodin limits of Woodin cardinals?

(3) Is the Mouse Set Conjecture true?

(4) Does Con(PFA) imply Con(Woodin limit of Woodins)?

Some basic open problems

By now, philosophy has far outrun the math. Some problems:

(1) How iterable is V? Are there iterable pure extender models having superstrongs? Measurable Woodin cardinals? Supercompacts?

(2) Are there hod mice with Woodin limits of Woodin cardinals?

(3) Is the Mouse Set Conjecture true?

(4) Does Con(PFA) imply Con(Woodin limit of Woodins)?Con(supercompacts)?

Some basic open problems

By now, philosophy has far outrun the math. Some problems:

(1) How iterable is V? Are there iterable pure extender models having superstrongs? Measurable Woodin cardinals? Supercompacts?

(2) Are there hod mice with Woodin limits of Woodin cardinals?

(3) Is the Mouse Set Conjecture true?

(4) Does Con(PFA) imply Con(Woodin limit of Woodins)?Con(supercompacts)?

(5) And ...

What's up there?