DETERMINACY FROM STRONG REFLECTION

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Abstract. The Axiom of Determinacy holds in the inner model $L(\mathbb{R})$ assuming Martin's Maximum for partial orderings of size c.

1. Introduction

A theorem of Neeman gives a particularly elegant sufficient condition for a set B of reals to be determined: B is determined if there is a triple (M, τ, Σ) which *captures* B in the sense that M is a model of a sufficient fragment of set theory, τ is a forcing term in M with respect to the collapse of some Woodin cardinal δ of M to be countable, and Σ is an $\omega + 1$-iteration strategy for M such that

$$B \cap N[g] = i(\tau)_g,$$

whenever $i : M \to N$ is an iteration map by Σ, and g is generic over N for the collapse of $i(\delta)$. The core model induction, subject of the forthcoming book [14], is a method pioneered by Woodin for constructing such triples (M, τ, Σ) by induction on the complexity of the set B. It seems to be the only generally applicable method for making fine consistency strength calculations above the level of one Woodin cardinal. We employ this method here to establish that the Axiom of Determinacy holds in the inner model $L(\mathbb{R})$ from consequences of the maximal forcing axiom $\mathsf{MM}(c)$, or Martin's Maximum for partial orderings of size c. The particular consequences we use are the saturation of the nonstationary ideal on ω_1, and the simultaneous reflection principle $\mathsf{WRP}_{(2)}(\omega_2)$ asserting that for any stationary subsets S and T of $[\omega_2]^\omega$ there is an ordinal $\delta < \omega_2$ so that $S \cap [\delta]^\omega$ and $T \cap [\delta]^\omega$ are both stationary in $[\delta]^\omega$.

Main Theorem. $\mathsf{WRP}_{(2)}(\omega_2)$ *plus NS saturated implies* $\mathsf{AD}^{L(\mathbb{R})}$.

Corollary. $\mathsf{MM}(c)$ *implies* $\mathsf{AD}^{L(\mathbb{R})}$.

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This theorem, obtained in late 2000, builds on Woodin's proof of PD from the same hypotheses (9.85 of [22]), and represents the first proof of the consistency of the Axiom of Determinacy from Forcing Axioms. The first author subsequently obtained the same conclusion in [19] from a single failure of square (and hence from PFA) building on Woodin's theorem that PFA together with an inaccessible gives AD in the Solovay model. Unlike that proof, which relies on covering lemmas to produce the models required for the induction step, we use the generic embedding derived from the saturated ideal. This has its precedents in the first author's proof of $\underset{\sim}{\Delta}^1_2$ determinacy from a presaturated ideal on ω_1 together with a measurable cardinal, and in Woodin's proof, via the core model induction, of $\mathsf{AD}^{L(\mathbb{R})}$ from an ω_1-dense ideal on ω_1.

While our theorem represents the best known lower bound for the consistency strength of $\mathsf{MM}(c)$, this principle is believed to be much stronger. In Chapter 6 we discuss some results suggesting that the arguments here cannot take us much farther than $\mathsf{AD}^{L(\mathbb{R})}$, and some extensions of the main theorem which could plausibly yield an equiconsistency result at the level of ω^2 Woodin cardinals from modified hypotheses.

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2. Framework of the induction

Let us recall some terminology from [19]

Definition 1. Let $U \subseteq \mathbb{R}$, and $k < \omega$. Let N be countable and transitive, and suppose $\delta_0, ..., \delta_k, S$, and T are such that

(a) $N \models \mathsf{ZFC} \wedge \delta_0 < ... < \delta_k$ are Woodin cardinals,
(b) $N \models S, T$ are trees which project to complements after the collapse of δ_k to be countable, and
(c) there is an ω_1+1-iteration strategy Σ for N such that whenever $i\colon N \to P$ is an iteration map by Σ and P is countable, then $p[i(S)] \subseteq U$ and $p[i(T)] \subseteq \mathbb{R} \setminus U$.

Then we say that N is a *coarse* (k, U)*-Woodin mouse*, as witnessed by $S, T, \Sigma, \delta_0, ..., \delta_k$.

Definition 2. W^*_α denotes the following assertion. If $U \subseteq \mathbb{R}$, and there are scales $\vec{\phi}$ and $\vec{\psi}$ on U and $\mathbb{R}\backslash U$ respectively such that $\vec{\phi}^*, \vec{\psi}^* \in J_\alpha(\mathbb{R})$, where $\vec{\phi}^*$ and $\vec{\psi}^*$ are the sequences of prewellorders associated to the scales, then for all $k < \omega$ and $x \in \mathbb{R}$ there are N, Σ such that

(1) $x \in N$, and N is a coarse (k, U)-Woodin mouse, as witnessed by Σ, and
(2) $\Sigma \restriction \mathrm{HC} \in J_\alpha(\mathbb{R})$.

Our core model induction will show that

$$V[g] \models \forall \alpha\ W^*_\alpha,$$

whenever $g \subset Col(\omega, \omega_1)$ is V-generic. From W^*_α we get a version of mouse capturing by fine-structural mice. Let us recall the relevant definitions from [19]. To any Σ_1 formula $\theta(v)$ we associate formulae $\theta^k(v)$ for $k \in \omega$, such that θ^k is Σ_k, and for any γ and any real x,

$$J_{\gamma+1}(\mathbb{R}) \models \theta[x] \Leftrightarrow \exists k < \omega J_\gamma(\mathbb{R}) \models \theta^k[x].$$

Our fine-structural witnesses are as follows.

Definition 3. Suppose $\theta(v)$ is a Σ_1 formula (in the language of set theory expanded by a name for $\mathbb{R}$), and z is a real; then a (θ, z)-witnesss is an ω-sound, $(\omega, \omega_1, \omega_1 + 1)$-iterable z-mouse $\mathcal{N}$ in which there are $\delta_0 < ... < \delta_9$, S, and T such that $\mathcal{N}$ satisfies the formulae expressing

(a) ZFC,
(b) $\delta_0, ..., \delta_9$ are Woodin,
(c) S and T are trees on some $\omega \times \eta$ which are absolutely complementing in $V^{Col(\omega,\delta_9)}$, and
(d) For some $k < \omega$, $p[T]$ is the Σ_{k+3}-theory (in the language with names for each real) of $J_\gamma(\mathbb{R})$, where γ is least such that $J_\gamma(\mathbb{R}) \models \theta^k[z]$.

Definition 4. W_α is the assertion: if $\theta(v)$ is Σ_1, $z \in \mathbb{R}$, and $J_\alpha(\mathbb{R}) \models \theta[z]$, then there is a (θ, z)-witness $\mathcal{N}$ whose associated iteration strategy, when restricted to countable iteration trees, is in $J_\alpha(\mathbb{R})$.

We have

Lemma 5. *Assume W^*_α holds; then*

(a) *$J_\alpha(\mathbb{R}) \models AD$, and*
(b) *W_α holds if α is a limit ordinal.*

See [19] for a proof of (b), which is essentially Woodin's mouse set theorem for $L(\mathbb{R})$. Part (a) is an easy exercise for our intended reader.[1] We note that W_α easily implies other forms of capturing by fine-structural mice, and in particular:

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[1]It follows directly from Neeman's theorem in [11], but one doesn't need that much firepower. The results of Martin-Steel [8], together with Woodin's genericity iterations (see [18]), give it easily.

Lemma 6. *Assume W_α holds; then if a is countable transitive, and $b \subseteq a$, b is ordinal definable from parameters in $a \cup \{a\}$ over some $J_\gamma(\mathbb{R})$, where $\gamma < \alpha$, then there is a a-premouse $\mathcal{M}$ such that $b \in \mathcal{M}$, and $J_\alpha(\mathbb{R}) \models \mathcal{M}$ is ω_1-iterable.*

We shall be proving that W^*_α holds in $V[g]$, by induction on α. Clearly, the only stages which matter are the *critical* ones, where

Definition 7. An ordinal β is *critical* just in case there is some set $U \subseteq \mathbb{R}$ such that U and $\mathbb{R} \setminus U$ admit scales in $J_{\beta+1}(\mathbb{R})$, but U admits no scale in $J_\beta(\mathbb{R})$.

Once again, we are identifying a scale with the *sequence* of its prewellorderings here. Clearly, we need only show that $W^*_{\beta+1}$ holds whenever β is critical, in order to conclude that W^*_α holds for all α. It follows from [20] that if β is critical, then $\beta + 1$ is critical. Moreover, if β is a limit of critical ordinals, then β is critical if and only if $J_\beta(\mathbb{R})$ is not an admissible set. Letting β be critical, we then have the following possibilities

1. $\beta = \eta + 1$, for some critical η;
2. β is a limit of critical ordinals, and either
   - (a) $cof(\beta) = \omega$, or
   - (b) $cof(\beta) > \omega$, but $J_\beta(\mathbb{R})$ is not admissible;
3. $\alpha = \sup(\{\eta < \beta \mid \eta \text{ is critical }\})$ is such that $\alpha < \beta$, and either
   - (a) $[\alpha, \beta]$ is a Σ_1 gap, or
   - (b) $\beta - 1$ exists, and $[\alpha, \beta - 1]$ is a Σ_1 gap.

We shall call (3) the *admissible case*, because it corresponds precisely to crossing a Σ_1 gap whose initial ordinal is admissible. In our proof of $W^*_{\beta+1}$, we get the capturing mice we need in $V[g][G]$, where $G \subset (P(\omega_1)/NS)^V$ is generic over $V[g]$. We then use an inductively maintained resemblance between $V[g]$ and $V[g][G]$ to find these mice in $V[g]$. This leads us to a second induction hypothesis which we denote I_α.

> Whenever $G \subset (P(\omega_1)/NS)^V$ is $V[g]$-generic, there is a Σ_1 embedding
> $$\pi : J_\alpha(\mathbb{R})^{V[g]} \to J_\alpha(\mathbb{R})^{V[G][g]}$$
> such that $\pi \restriction \omega\alpha$ is the identity.

This hypothesis is not used as an input in the arguments of the admissible case but is produced as an output. On the other hand, it is used in the inadmissible case. Since it can be argued that W^*_α implies I_α for such limit ordinals α, we could avoid carrying it along. I_α originates in [23] where it is part of a proof that the saturation of NS and

$\mathsf{WRP}_{(2)}(\omega_2)$ imply W^*_α for α the first admissible over the reals. Our argument here follows the overall structure of [23] pretty closely. What we add are some techniques for getting past admissible ordinals using hybrid strategy mice. These techniques were also used in [19].

3. The Projective Case

We begin with the following result, due to Woodin and implicit in [22] (see Thm. 9.82 for example), which illustrates how our hypotheses work in conjunction. It uses his theorem that Continuum Hypothesis is incompatible with the saturation of NS in the presence of a measurable cardinal (3.17 of [22]).

Lemma 8. *Assume* $\mathsf{WRP}_{(2)}(\omega_2)$ *and* NS *saturated. Then*

$$2^\omega = 2^{\omega_1} = \underset{\sim}{\delta}^1_2 = \omega_2.$$

Proof. A theorem of Todorcevic (see Thm 6.4 of [6]) gives $2^\omega \leq \omega_2$ under $\mathsf{WRP}(\omega_2)$. The idea is that there is always an injection from 2^ω into any club subset C of $[\omega_2]^\omega$, and under $\mathsf{WRP}(\omega_2)$ there is such a club of size ω_2, namely

$$C = \bigcup_{\delta<\omega_2} C_\delta$$

where each C_δ be a club of size ω_1 in $[\delta]^\omega$. Now, NS saturated gives closure of $P(\omega_1)$ under the dagger operation and $\mathsf{WRP}_{(2)}(\omega_2)$ lifts this closure to $P(\omega_2)$ as in Lemma 14 below. We construct a function $B : \omega_2 \to P(\omega_1)$ as follows. Let $B(0) \subset \omega_1$ be such that $\omega_1^{L[B]} = \omega_1$. Given $B \restriction \gamma$ let $(X_\xi \mid \xi < \omega_1)$ enumerate those sets $X \in (B \restriction \gamma)^\dagger \cap NS$ which are stationary in $(B \restriction \gamma)^\dagger$. Let $B(\gamma)$ code a sequence of clubs disjoint from the X_ξ as well as a surjection from ω_1 to γ. It follows easily that $B^\dagger$ thinks that NS is saturated and that $(\omega_2)^{B^\dagger} = \omega_2$. Since $B^\dagger$ has a measurable cardinal it follows from 3.17 of [22] that $\underset{\sim}{\delta}^1_2 = \omega_2$ in $B^\dagger$ and thus $2^\omega = \omega_2$ in V as desired. Saturation of the nonstationary ideal implies that $j(\omega_2) < \omega_3^V$ where j is the map into the generic ultrapower. This is because $j(\omega_1) = \omega_2$, ω_2 and ω_3 remain cardinals, and j is continuous at ω_2. It follows that $2^{\omega_1} = \omega_2$. To see this note that $P(\omega_1)^V$ is a subset of the generic ultrapower and hence CH holds in the extension. It follows that there is an injection from $P(\omega_1)^V$ to ω_2^V in the extension and so $2^{\omega_1} = \omega_2$ must hold in V. □

Our main goal in this section to give a proof of Woodin's theorem (9.85 of [22]) that our hypotheses imply Projective Determinacy. This is driven by a theorem of Steel (Thm. 7.1 of [16]) on the nonexistence of the core model in suitable environments in which there is a generic almost-huge embedding. Only a little extra work is required to prove the theorem with the NS saturated hypothesis weakened to the existence of a presaturated ideal on ω_1 so we will do this instead.

Theorem 9. *(Steel) Assume there is a measurable cardinal and a presaturated ideal on ω_1. Then $\underset{\sim}{\Delta}^1_2$ determinacy holds.*

Theorem 10. *(Woodin) Assume $\mathsf{WRP}_{(2)}(\omega_2)$ and NS saturated. Then* PD *holds and continues to hold in the universe after ω_2 is collapsed.*

Woodin's argument shows inductively that $H(\omega_3)$ is closed under the $M_n^\#$ operation for each $n < \omega$. Core Model Theory together with NS saturated and the induction hypothesis is used to close $H(\omega_1)$ under the next operation. That is, if N is the minimal $M_n^\#$ closed model containing $P(\omega_1)$, and $N^\#$ exists, then the argument of Theorem 9 shows that $M_{n+1}^\#(x)$ exists for every $x \in H(\omega_1)$. The generic ultrapower map lifts this closure to $H(\omega_2)$ in the following way. If

$$M \simeq \mathrm{Ult}(V, G) \subset V[G]$$

is the generic ultrapower and $a \in H(\omega_2)^V$ then $a \in M$ so $M \models M_{n+1}^\#(a)$ exists. By a mutual genericity argument using closure of $H(\omega_3)$ under the $M_n^\#$ operation we will conclude that $M_{n+1}^\#(a) \in V$. $\mathsf{WRP}_{(2)}(\omega_2)$ further extends this closure to $H(\omega_3)$. Here we will fill in this outline to produce mice $M_n^{\mathcal{M},\#}$ which have an active top extender, n Woodin cardinals, and which are closed under a given first order mouse operator $\mathcal{M}$. In the case $\mathcal{M} = \emptyset$, this will give a proof of PD.

For a transitive set a, an a-premouse M is countably iterable if whenever $\pi : N \to M$ is elementary with N countable then N is $\omega_1 + 1$ iterable. For a transitive set a let $o(a)$ denote the smallest ordinal not in a. Any two sound countably iterable a-premice which project to $o(a)$ are comparable (see [18]). The lower part closure of a is defined as the union of all a-premice N which are countably iterable, sound, and satisfy $\rho_\omega(M) = o(a)$. $L_p(a)$ is a countably iterable a-premouse in its own right.

Definition 11. For a set $a \in H(\theta)$ a mouse operator over a on $H(\theta)$ is an operation associating to each transitive set $b \in H(\theta)$ the least level $\mathcal{M}(b)$ of $Lp(b)$ such that $\mathcal{M}(b) \models \phi(a, b)$ for some fixed formula ϕ. We assume that a is always coded into b in such a way that $a \in L[b]$.

For any mouse operator $\mathcal{M}$ and set b in its domain there is a corresponding minimal $\mathcal{M}$-closed model $L^{\mathcal{M}}(b)$. This is obtained by concatenating the $\vec{E}$ predicates. A condensation argument shows that $L^{\mathcal{M}}(b)$ is a bone-fide b-mouse. An embedding from $L^{\mathcal{M}}(b)$ to itself with critical point above b gives rise to a mouse $\mathcal{M}^{\#}(b)$ with corresponding operator $\mathcal{M}^{\#}$. We let $\mathcal{M}^*$ denote the operator $(\mathcal{M}^{\#})^{\#}$. That is, $\mathcal{M}^*(b)$ is minimal with respect to having an active top extender and being closed under the $\mathcal{M}^{\#}$ operation. There is also an associated operator $M_n^{\mathcal{M},\#}$ which associates to a set b the minimal $\mathcal{M}$-closed b-mouse with n Woodin cardinals and an active top extender. There is a first order formula which expresses this in each case. We refer the reader to [19] and especially [14] for a more extended discussion of these concepts.

Definition 12. A mouse operator $\mathcal{M}$ defined on $H(\theta)$ relativizes well if whenever $a, b \in H(\theta)$ are transitive with $a \in b$ then $\mathcal{M}(a)$ belongs to all transitive models of a sufficiently large fragment of ZFC that $\mathcal{M}(b)$ does. We say $\mathcal{M}$ determines itself on generic extensions if whenever $\mathcal{P} \in H(\theta)$ is a poset and G is V-generic then $H(\theta)^{V[G]}$ is closed under $\mathcal{M}$.

Definition 13. For regular cardinals $\kappa < \lambda$ we say that Mouse Reflection at (κ, λ) holds if for every $a \in H(\kappa)$ and every mouse operator over a which is total on $H(\kappa)$ is also total on $H(\lambda)$. If $\lambda = \kappa^+$ we say that Mouse Reflection holds at κ.

A theme of this paper is that simultaneous reflection can be used to lift closure under certain operations from $P(\omega_1)$ to $P(\omega_2)$. In [22] Woodin gives a proof that under $\mathsf{WRP}_{(2)}(\omega_2)$, closure of $P(\omega_1)$ under sharps entails closure of $P(\omega_2)$ under sharps, and his proof of Theorem 10 involved analogous arguments for the $M_n^{\#}$ operation. The following is a straighforward generalization to first order mouse operators. We also refer the reader to [24] where it is shown that ω_1-Universally Baire self-justifying systems are ω_2-Universally Baire under $\mathsf{WRP}_{(2)}(\omega_2)$.

Lemma 14. $\mathsf{WRP}_{(2)}(\omega_2)$ *implies Mouse Reflection at* ω_2.

Proof. Let $\mathcal{M}$ be a first order mouse operator and suppose $H(\omega_2)$ is closed under $\mathcal{M}$. We assume that $\mathcal{M}$ is a mouse operator over $\emptyset$ for simplicity. Fix a transitive set a in $H(\omega_3)$. Because $\rho_\omega^{\mathcal{M}(b)} = o(b)$ for any set in the domain of $\mathcal{M}$ we may regard any such $\mathcal{M}(b)$ as a subset of $b^{<\omega}$. We will abuse notation and let $\mathcal{M}(b)$ refer to this code as well. When σ and A are sets of ordinals, we let $A^\sigma = \pi_\sigma[A \cap \sigma]$ where $\pi_\sigma \to otp(\sigma)$ is an order isomorphism. By fixing a bijection $f : \omega_2 \to a$ we may unambiguously speak of a^σ for $\sigma \in [\omega_2]^\omega$. For $t \in a^\omega$ we define a candidate $\mathcal{P}$ for $\mathcal{M}(a)$ by $t \in \mathcal{P}$ if and only if S_t contains a club where

$$S_t = \{\sigma \in [\omega_2]^\omega \mid t^\sigma \in \mathcal{M}(a^\sigma)\}.$$

We first show that the sets S_t are measured by the club filter. Otherwise there is such a t and a $\delta < \omega_2$ such that $S_t \cap [\delta]^\omega$ is stationary and costationary in $[\delta]^\omega$. Suppose without loss of generality that $t \in \mathcal{M}(a \cap \delta)$. Let $X \prec H(\theta)$ be countable and contain δ, t and $\mathcal{M}(a \cap \delta)$. We may assume that $\sigma = X \cap \delta \notin S_t$. Let $\pi : X \to H$ be the transitivization map. Thus

$$\pi(\mathcal{M}(a \cap \delta)) = \mathcal{M}(a^\sigma)$$

and so $t^\sigma \in \mathcal{M}(a^\sigma)$ which is a contradiction. It follows by a pressing down argument that for a club of countable elementary $X \prec H(\theta)$ with transitive collapse $\pi : X \to H$ we have $\pi(\mathcal{P}) = \mathcal{M}(\pi(a))$. To finish the argument suppose $j : \bar{\mathcal{P}} \to \mathcal{P}$ is fully elementary with $\bar{\mathcal{P}}$ countable. Let π and H be as above with j in the range of π. Thus

$$\pi(j) : \bar{\mathcal{P}} \to \pi(\mathcal{P}) = \mathcal{M}(\pi(a))$$

is elementary and we conclude that $\bar{\mathcal{P}}$ is $\omega_1 + 1$ iterable. It follows that $\mathcal{P} = \mathcal{M}(a)$ as desired. □

Only a little extra work is required to prove Theorem 3 with the NS saturated assumption weakened to the existence of a presaturated ideal on ω_1. The only complication is the potential failure of $2^{\omega_1} = \omega_2$. If we were to assume $2^{\omega_1} = \omega_2$ and $I = NS$ then any B with $P(\omega_1) \in L[B]$ would suffice for the following lemma.

Lemma 15. *Suppose I is a presaturated ideal on ω_1 and $\mathcal{M}$ is a mouse operator defined on $H(\omega_3)$. Suppose that $H(\omega_2)$ is closed under $\mathcal{M}^\#$. Then there is a $B \subset \omega_2$ such that*

(1) *$H(\omega_2)^{L^{\mathcal{M}}(B)}$ is fully elementary in $H(\omega_2)$*
(2) *$\bar{I} = I \cap L^{\mathcal{M}}(B) \in L^{\mathcal{M}}(B)$ and*
(3) *$L^{\mathcal{M}}(B) \models \bar{I}$ is a presaturated ideal on ω_1.*

Proof. We inductively fold in all of the necessary data to B which we regard as a function from ω_2 to $P(\omega_1)$. Let $B(0)$ be such that $\omega_1^{L[B(0)]} = \omega_1$. Given $B \restriction \delta$ let $B(\delta)$ code W_δ and I_δ where I_δ in turn codes

$$I \cap \mathcal{M}^\#(B \restriction \delta)$$

and $W_\delta \in [P(\omega_1)]^{\omega_1}$ has the following property. If $a \in \mathcal{M}^\#(B \restriction \delta)$ and $\phi(x, y)$ is a formula of the language of set theory so that

$$H(\omega_2)^V \models \phi(b, a)$$

for some b then there is such a b coded (in some simple way) by an element of W_δ. We set $\bar{I}$ to the union of the sets I_δ. □

Appropriate closure under $\mathcal{M}^\#$ is required to use π_{NS} to lift closure of $M_1^{\mathcal{M},\#}$ from $P(\omega)$ to $P(\omega_1)$. This would not be the case if $P(\omega_1)/NS$ were homogeneous. Closure of $P(\omega)$ under the $M_1^{\mathcal{M},\#}$ operation comes by way of 7.1 of [16] and the K^J-existence dichotomy of [14].

Lemma 16. *Assume there is a presaturated ideal on ω_1. Suppose $H(\omega_3)$ is closed under $\mathcal{M}^*$ for some mouse operator $\mathcal{M}$ which relativizes well. Then $H(\omega_1)$ is closed under the operator $M_1^{\mathcal{M},\#}$.*

Proof. Fix $a \in H(\omega_1)$. We let B be as in the lemma above with respect to the operator $\mathcal{M}^\#$. Thus $\mathcal{M}^*(B)$ exists and $\mathcal{M}^*(B) \models I$ is a presaturated ideal on ω_1 for some $I \in \mathcal{M}^*(B)$. We may of course assume that $a \in \mathcal{M}^*(B)$ by adding it into $B(0)$. Working inside $\mathcal{M}^*(B)$ we see that $K(a)$ cannot exist by the argument of Theorem 9. It follows from the K^J-dichotomy of [14] that $M_1^{\mathcal{M},\#}(a)$ exists in $\mathcal{M}^*(B)$ and hence in V by clause (3) of Lemma 15. □

Lemma 17. *Assume there is a presaturated ideal on ω_1. Suppose $H(\omega_3)$ is closed under $\mathcal{M}^*$ for some mouse operator $\mathcal{M}$ which relativizes well and determines itself on generic extension. Then $M_1^{\mathcal{M},\#}(a)$ exists for every $a \in H(\omega_2)$.*

Proof. Fix $a \in H(\omega_2)$ and let B be as in the lemma above with $a \in \mathcal{M}^*(B)$. Thus $\mathcal{M}^*(B) \models I$ is a presaturated ideal on ω_1 for some $I \in \mathcal{M}^*(B)$ and also $\mathcal{M}^*(B)$ thinks that $H(\omega_1)$ is closed under the $M_1^{\mathcal{M},\#}$ operation by Lemma 16. Let

$$\pi : \mathcal{M}^*(B) \to N \subset \mathcal{M}^*(B)[G]$$

be the generic ultrapower derived from a $\mathcal{M}^*(B)$-generic $G \subset (P(\omega_1) \cap \mathcal{M}^*(B))/I$. Thus $N \models M_1^{\mathcal{M},\#}(a)$ exists. Call this structure $\mathcal{P}$. The iteration strategy for $\mathcal{P}$ uses the $\mathcal{M}^\#$ operator to identify the correct well-founded branch at limit stages. By closure of N inside $\mathcal{M}^*(B)[G]$, $\mathcal{P}$ remains ω_1-iterable in $\mathcal{M}^*(B)[G]$. Let us assume that $\mathcal{T}$ is a tree of length ω_1 on $\mathcal{P}$ played according to this strategy. Inside $\mathcal{M}^*(B)[G]$ let $j : H \to H(\theta)$ with θ large enough, H countable and transitive, and $ran(j)$ containing $\mathcal{T}, \mathcal{P}$ and $\mathcal{M}^\#(M(\mathcal{T}))$ where $M(\mathcal{T})$ is the common part of the tree T. Clearly $j^{-1}(P) = P$. Since

$$j^{-1}(\mathcal{M}^\#(M(\mathcal{T}))) = \mathcal{M}^\#(M(j^{-1}(\mathcal{T}))$$

and since $j^{-1}(\mathcal{T})$ is a tree built according to the iteration strategy of $\mathcal{P}$ we conclude by a simple absoluteness argument that H must see the correct branch through $j^{-1}(\mathcal{T})$. Now we argue that N is independent of the generic. It suffices to show that $\mathcal{P}$ remains iterable in $V[G][\bar{G}]$ where $\bar{G}$ is $V[\bar{G}]$ generic for $(P(\omega_1)/I)^V$. This follows by a similar absoluteness argument using the closure under the $\mathcal{M}^\#$ operator. □

We can now prove Woodin's Theorem in the setting of first order mouse operators.

Lemma 18. *Assume there is a presaturated ideal on ω_1 and $\mathsf{WRP}_{(2)}(\omega_2)$ holds. Suppose $\mathcal{M}$ is a mouse operator defined on $H(\omega_2)$ which relativizes well and determines itself on generic extensions. Then $M_n^{\mathcal{M},*}$ is total on $H(\omega_3)$ for every $n < \omega$.*

Proof. This is proven by induction on $n < \omega$. By Lemma 14 we get closure of $H(\omega_3)$ under M. Let

$$\pi : V \to M \subset V[G]$$

denote the generic ultrapower. Fix $x \in H(\omega_1)$. It is easily seen that $\pi(L^{\mathcal{M}}[x])$ agrees with $L^{\mathcal{M}}[x]$ through ω_2^V. It follows that ω_2^V is inaccessible in this structure. Let μ be the measure on $P(\omega_1^V) \cap L^{\mathcal{M}}[x]$. By standard arguments we can fashion $\mathcal{M}^{\#}(x)$ out of this material the independence of the generic argument from Lemma 17 now shows that $\mathcal{M}^{\#}(x) \in V$. We now work the argument of 17 again to get closure of $H(\omega_2)$ under the $\mathcal{M}^{\#}$ operation, lift this to $H(\omega_3)$ by Lemma 14, and then repeat the steps we have just taken again to get closure of $H(\omega_2)$ under the $\mathcal{M}^*$ operation. Now suppose $H(\omega_3)$ is closed under the $M_n^{\mathcal{M},*}$ operation for an arbitrary $n < \omega$. Lemma 16 gives closure of $H(\omega_1)$ under the $M_{n+1}^{\mathcal{M},\#}$ operation and Lemmas 17 and 14 lift this closure to $H(\omega_3)$. An argument very similar to that of the base case gives closure under $M_{n+1}^{\mathcal{M},*}$ and completes the induction step. □

The structures produced above will give PD in V and in the universe after collapsing ω_2. To motivate some later arguments however, we now use descriptive set-theoretic methods to argue from our hypotheses that having PD in V gives PD in the universe after collapsing ω_2. This will use the Third Periodicity theorem of Moschovakis.

Lemma 19. *(Moschovakis; 6E of [14]) Suppose Γ as adequate pointclass and every set in Γ admits a Γ-semiscale. Suppose all Δ games are determined. Let $A \in \Delta$. Then a winning strategy for the game associated to A can be can be computed from the knowledge of which player wins each of a sequence of Δ games associated to the norms of the Γ-semiscales on A and its complement.*

We will need that PD is preserved by Cohen forcing. The key ideas of the following proof can be found in [2]. It works just as well for $Col(\omega, \kappa)$ if the property of Baire is strengthened to the κ-Universal Baire property.

Lemma 20. *Assume all projective sets have the property of Baire and that every projective subset of $\mathbb{R} \times \mathbb{R}$ has a projective uniformizing function. Suppose $g \subset Col(\omega, \omega)$ is V-generic. Then $\mathbb{R}^V \prec \mathbb{R}^{V[g]}$.*

Proof. Since projective sets are closed under continuous preimages it follows from Thm. 2.1 of [6] that projective sets are ω-universally Baire. Thus for each $n \geq 1$ there are trees S_n, T_n with $p[S_n]$ the universal Σ^1_n set of reals (which we denote by A_n in whichever model it is computed) and

$$p[T_n] = \mathbb{R} \setminus p[S_n]$$

whenever $g \subset Col(\omega, \omega)$ is V-generic. We argue by induction on $n \geq 2$ that

$$(p[S_n] = A_n)^{V[g]}$$

for any generic $g \subset Col(\omega, \omega)$. The result follows by an elementary submodel argument. Fix n and suppose τ is standard term for a real so that

$$\Vdash_{Col(\omega,\omega)} \tau \in p[S_n].$$

We can assume that

$$p[S_n] = \{x \mid \exists y \phi(x, y)\}$$

where ϕ is Π^1_{n-1}. A submodel argument gives a comeager set of $h \in \omega^\omega$ such that $\exists y \phi(\tau_h, y)$ holds in V. Using our uniformization hypothesis we can assemble a term τ^* so that $\phi(\tau_h, \tau^*_h)$ similarly holds on a comeager set of h. To see this let f uniformize

$$\{(x, y) \mid \phi(x, y)\}$$

in that $\phi(x, f(x))$ holds whenever there is any y such that $\phi(x, y)$ holds. The (partial) function g defined by $g(h) = f(\tau_h)$ is defined and continuous on a comeager set so there is a standard term τ^* such that $tau^*_h = g(h)$, and hence $\phi(\tau_h, \tau^*_h)$ holds, for a comeager many h. From our induction hypothesis we now get

$$(p[S_n] \subseteq A_n)^{V[g]}$$

for any generic $g \subset Col(\omega, \omega)$. Now, again using our induction hypothesis we get a tree S^*_n which always projects to A_n in any $V[g]$. It follows that

$$A_n \subseteq p[S_n])^{V[g]}$$

as desired, otherwise $p[S^*_n] \cap p[T_n]$ would be nonempty in V. □

If PD holds in $V[g]$ as below then $\mathsf{WRP}_{(2)}(\omega_2)$ suffices for the conclusion as in 9.84 of [22]. The harder part is getting PD in $V[g]$.

Lemma 21. *Assume $WRP_{(2)}(\omega_2)$ and NS saturated. Suppose PD holds. Then PD holds in the universe after collapsing ω_2 and*

$$\mathbb{R}^{V[g]} \prec \mathbb{R}^{V[G][g]}$$

whenever $g \subset Col(\omega, \omega_1^V)$ is $V[G]$-generic and $G \subset P(\omega_1)/NS$ is V-generic.

Proof. Let $\Lambda = \bigcup_{n<\omega} \omega n \underset{\sim}{\Pi}^1_1$ (see [15]) and let $\mathcal{G}$ denote the game quantifier (see [10]). For $n < \omega$, any set in the pointclass $\mathcal{G}^n\Lambda$ has a scale whose norms also belong to the pointclass. Further, every game in $\mathcal{G}^n\Lambda$ has a winning strategy which can be computed from the knowledge of which player wins a certain sequence of $\mathcal{G}^n\Lambda$ games. Also, assuming determinacy of $\mathcal{G}^{n-1}\Lambda$ games,

$$\underset{\sim}{\Sigma}^1_{n+1} \cup \underset{\sim}{\Pi}^1_{n+1} \subseteq \mathcal{G}^n\Lambda \subseteq \underset{\sim}{\Delta}^1_{n+2}.$$

Now let $G \subset P(\omega_1)/NS$, $g_1 \subset Col(\omega, \omega_1)$, and $g_2 \subset Col(\omega, \omega_2)$ be such that G is V-generic, g_1 is generic over $V[G]$ and g_2 is generic over $V[G][g_1]$. PD holds in $V[G]$ using the generic embedding. Since g_1 is Cohen generic over $V[G]$, the argument of Lemma 20 below shows that PD holds in $V[G][g_1]$ and that $V[G] \prec_{\Sigma^1_\omega} V[G][g_1]$. We will show by induction on $n \in \omega$ that

(a) $\mathcal{G}^n\Lambda$-determinacy holds in $V[g_1]$
(b) $\mathcal{G}^n\Lambda$-determinacy holds in $V[g_2]$
(c) $V[g_1] \prec_{\Sigma^1_{n+3}} V[G][g_1]$ and $V[G][g_1] \prec_{\Sigma^1_{n+3}} V[G][g_1][g_2]$.

Fix $n < \omega$ and assume the conditions above hold for every $m \leq n$. Let $A \in \mathcal{G}^n\Lambda$ in $V[g_1]$. The the same definition gives rise to a set in the $\mathcal{G}^n\Lambda$ of $V[G][g_1]$ which we also denote by A. By Third Periodicity, the definable winning strategy for A in this latter model can be computed from a list of the winners of a certain sequence of games A_k which are also in $\mathcal{G}^n\Lambda$. We must see that this data is independent of G. Otherwise we may assume there are $G_0, G_1 \subset P(\omega_1)/NS$ which are $V[g_1]$-generic and $\tau_i \in V[G_i][g_1]$ for each i such that

$$V[g_1][G_0] \models \tau_0 \text{ is winning for I in the game } G(A_k)$$

and

$$V[g_1][G_1] \models \tau_1 \text{ is winning for II in the game } G(A_k).$$

These statements, $\phi(\tau_0)$ and $\phi(\tau_1)$ are both Π^1_{n+3} and hence hold in $V[g_1][G_0][g_2]$ and $V[g_1][G_1][g_2]$ respectively. This is a contradiction as these models can be reorganized as homogeneous extensions of $V[g_1]$ by $Col(\omega, \omega_2)$. This $\mathcal{G}^n\Lambda$ determinacy holds in $V[g_1]$. Using a $WRP_{(2)}(\omega_2)$ argument we lift this to any $V[g_2]$. To see this fix a standard $Col(\omega, \omega_2)$ term for a real τ and a definition of a $\mathcal{G}^n\Lambda$ game $A(\tau)$. Note that $Col(\omega, \gamma)$ is isomorphic to Cohen forcing for any $\gamma < \omega_1$ and hence PD holds in $V^{Col(\omega,\gamma)}$ by Lemma 8. We will regard a strategy as a subset of $\omega^{<\omega}$. For $p \in Col(\omega, \omega_2)$ and $t \in \omega^{<\omega}$ we define $S_{p,t}$ to be the set of

$\sigma \in [\omega_2]^\omega$ such that

$p^\sigma \Vdash_{Col(\omega,otp(\sigma))} t$ belongs to the canonical $G(A(\tau^\sigma))$ winning strategy.

Similarly we define $T_{p,t}$ to be the set where t does not belong to the canonical winning strategy as above. We claim that $S_{p,t}$ and $T_{p,t}$ cannot both be stationary. Otherwise $WRP_{(2)}(\omega_2)$ gives a $\gamma < \omega_2$ above p so that they both reflect to $[\gamma]^\omega$. Let $\{\sigma_\xi \mid \xi < \omega_1\}$ be continuous and exhaustive in $[\gamma]^\omega$ and let $S = \{\xi \mid \sigma_\xi \in S_{p,t}\}$ and $T = \{\xi \mid \sigma_\xi \in T_{p,t}\}$. Let $G_S, G_T \subset P(\omega_1)/NS$ be generic over $V[g]$ where $g \subset Col(\omega, \gamma)$ is V-generic. Then t belongs to the canonical $G(A((\tau \restriction \gamma)_g))$ winning strategy in $V[G_S][g]$ and does not in $V[G_T][g]$. This contradicts induction hypothesis (c). It is now easily seen that the term $\bar{\tau}$, defined as (abusing notation) the set of (p, t) such that $S_{p,t}$ contains a club, has the property that for a club of σ and any generic $h \subset Col(\omega, otp(\sigma))$ the model $V[h]$ thinks that $\bar{\tau}^\sigma_h$ is the canonical winning strategy for the game $G(A(\tau^\sigma_h))$. We claim that $\bar{\tau}$ is winning for $G(A(\tau))$ in $V^{Col(\omega,\omega_2)}$. Otherwise we can find an elementary submodel X of a large enough $H(\theta)$ with transitive collapse $\pi : X \to H$ such that $\sigma = X \cap \omega_2$ is in the club referenced above and a V-generic $h \subset Col(\omega, otp(\sigma))$ such that $H[h]$ thinks that $\bar{\tau}^\sigma_h$ does not win $G(A(\tau^\sigma_h))$. We may assume that there are trees S, T such that $p[S] = A(\tau^\sigma_h)$ and $p[T] = \mathbb{R} \setminus A(\tau^\sigma_h)$ in $V[h]$ and such that

$$H[h] \models p[\pi(S)] = A(\tau^\sigma_h) \text{ and } p[\pi(T)] = \mathbb{R} \setminus A(\tau^\sigma_h).$$

Thus $H[g]$ thinks that there is a play by $\bar{\tau}^\sigma_h$ which belongs to $p[\pi(T)]$ and hence there really is in $V[h]$ giving the desired contradiction. This gives part (b) of the induction hypothesis. Part (c) follows using the trees of the new scales. □

Thus, under the hypotheses of Lemma 21, sentences of second order arithmetic are decided by $P(\omega_1)/NS$ with boolean value one. Recalling that $J_0(\mathbb{R}) = V_{\omega+1}$, another way to state the conclusion above is that an embedding

$$\pi : J_1(\mathbb{R})^{V[g]} \to J_1(\mathbb{R})^{V[G][g]}$$

exists which is Σ_1 elementary and fixes ordinals and reals.

4. The Inadmissible Case

This section has two subsections. In the first we present some relevant aspects of the fine structure of $L(\mathbb{R})$ extracted from [20], and prove some local versions of a well-known theorem of Woodin to the effect that $\mathsf{AD}^{L(\mathbb{R})}$ is preserved by Cohen forcing. In the second we handle the induction step in the inadmissible case under the assumption that α is an inadmissible limit ordinal which begins a gap in the universe after collapsing ω_1. First we give some background information, and discuss the strategy of [29], some aspects of which are retained in the current proof of the Main Theorem. In particular we explain the origin of the auxiliary hypothesis I_α which will figure into the argument of 4.2.

Recall that in section 3 we proved PD by a "cycling" argument, first closing $H(\omega_1)$ under a mouse operator, then lifting this closure to $H(\omega_2)$ and finally to $H(\omega_3)$. The thinking around the time of [23] was that the main obstacle to continuing this "cycling" argument through the levels of $L(\mathbb{R})$ was the lack of homogeneity of the forcing $P(\omega_1)/NS$ as it impacts the $H(\omega_1)$ to $H(\omega_2)$ step. To address this problem, we wanted to relate the $L(\mathbb{R})$ of the homogeneous extension by $Col(\omega, \omega_1)$ that of an extension by $P(\omega_1)/NS \times Col(\omega, \omega_1)$. This latter model $V^{P(\omega_1)/NS \times Col(\omega,\omega_1)}$ is a Cohen extension of $V^{P(\omega_1)/NS}$ and a *ccc* extension of $V^{Col(\omega,\omega_1)}$ if NS is saturated. It was believed that the existence of an embedding between the $L(\mathbb{R})$ of these models together with our hypotheses would yield $\mathsf{AD}^{L(\mathbb{R})}$. This weaker version of our main theorem is as follows.

> Assume $WRP_{(2)}(\omega_2)$ and NS saturated. Suppose that whenever $G \subset P(\omega_1)/NS$ is V-generic and $g \subset Col(\omega, \omega_1)$ is $V[G]$-generic there is an embedding
> $$\bar{\pi} : L(\mathbb{R})^{V[g]} \to L(\mathbb{R})^{V[G][g]}$$
> which is Σ_1 and fixes ordinals. Then AD holds in $L(\mathbb{R})$.

On the other hand, it was also known that if the induction were to succeed in proving AD in $L(\mathbb{R})$, and that this persists after collapsing ω_2, then we would have such an embedding. This is by results of Woodin (from [2] and [22]) and because ccc forcing does not change the lengths of Universally Baire prewellorderings by a result of Foreman and Magidor from [3]. The following is one way to make this precise.

Lemma 22. *Assume NS saturated and* $\mathsf{WRP}_{(2)}(\omega_2)$. *Assume* $\mathsf{AD}^{L(\mathbb{R})}$ *holds in* $V[g]$ *whenever* $g \subset Col(\omega, \omega_1)$ *is* V*-generic. Suppose* G *and* g *are the factors of generic filter on* $P(\omega_1)/NS \times Col(\omega, \omega_1)$. *Then there is a fully elementary embedding*
$$\pi : L(\mathbb{R})^{V[g]} \to L(\mathbb{R})^{V[G][g]}$$
and any such embedding satisfies $\pi \restriction \theta^{L(\mathbb{R})} = id$. *Moreover,* $AD^{L(\mathbb{R})}$ *holds in* V *and in the universe after collapsing* ω_2.

Proof. By Lemma 8 we have $2^{\omega_1} = \omega_2$. Thus from the perspective of $V[g]$ the algebra $\mathbb{B} = (P(\omega_1)/NS)^V$ has size ω_1. We claim it also satisfies the countable chain condition in $V[g]$. Otherwise there is a condition p which forces that some $\dot{f}$ enumerates an antichain of length ω_2^V. On cardinality grounds there must be a condition $q \leq p$ which decides ω_2 of the values of $\dot{f}$, a contradiction as two of these values must therefore be compatible by saturation of NS in V. Now, since $2^{\omega_1} = \omega_2$ and $P(\omega_2)$ is closed under sharps, we have $\mathbb{R}^\#$ in $V[g]$. By 9.83 of [28] then we have that $\mathbb{R}^\#$ exists and $L(\mathbb{R}) \models \mathsf{AD}$ in the universe after collapsing ω_2. This uses $\mathsf{WRP}_{(2)}(\omega_2)$. Now we may assume there is $h \subset Col(\omega, \omega_2)$ which is $V[G][g]$ generic and so that $V[G][g][h] = V[\bar{h}]$ for some $\bar{h} \subset Col(\omega, \omega_2)$. The argument of 5.2 of [2] now shows that there are definable trees S, T in $V[g]$ such that $p[S] = \mathbb{R}^\#$ and $p[T] = \mathbb{R} \setminus \mathbb{R}^\#$ in $V[\bar{h}]$. These trees belong to $V, V[g]$, and $V[G][g]$ by homogeneity and it can be argued that $p[S] = \mathbb{R}^\#$ in the sense of each model. Thus there is a fully elementary embedding

$$\pi : L(\mathbb{R})^{V[g]} \to L(\mathbb{R})^{V[G][g]}.$$

We also have that from the perspective of $V[g]$, every set of reals in $L(\mathbb{R})$ is ω_1-Universally Baire and hence $\mathbb{B}$-Universally Baire as $2^{\omega_1} = \omega_2$. Now suppose π is such an embedding and fix α less than the θ of $L(\mathbb{R})^{V[g]}$. There is a prewellordering $\preceq \in L(\mathbb{R})^{V[g]}$ (with associated equivalence relation $\simeq$) of length α in $V[g]$ and trees S, T such that

(1) $p[S] = \preceq$ in $V[g]$
(2) $p[S] = \pi(\preceq)$ in $V[G][g]$
(3) $p[S] = \mathbb{R} \setminus p[T]$ in $V[g][G]$.

Since $\pi(\alpha)$ is the length of $\pi(\preceq)$ we must show that $p[S]^{V[G][g]}$ has length α. We will adapt an argument of Foreman and Magidor to show that every real in $V[G][g]$ is equivalent in the sense of $p[S]$ to a real in $V[g]$. The result will follow. Note that clauses (2) and (3) holds as well if G were $\mathbb{B} \times \mathbb{B}$ generic. So suppose toward a contradiction that τ is a term for a real which is forced over $V[g]$ by some condition of $\mathbb{B}$ (which we suppress) to be inequivalent to every real in $V[g]$. We let τ_l and τ_r be the $\mathbb{B} \times \mathbb{B}$ terms for interpretation of τ by the left and right generic respectively. The following argument is found in the proof of Theorem 3.4 of [3].

Claim 23. There is no $p \in \mathbb{B}$ such that $(p, p) \Vdash^{V[g]}_{\mathbb{B} \times \mathbb{B}} \tau_l \simeq \tau_r$.

Proof. By $\tau_l \simeq \tau_r$ we mean $(\tau_l, \tau_r) \in p[S]$ and $(\tau_l, \tau_r) \in p[S]$. Suppose there is such a p. Then there is an elementary submodel of a sufficient rank initial segment of the universe containing p and generics G', G_0, G_1, G_2 such that $G' \subset \mathbb{B}$ is $V[g]$-generic, $G_0 \in V[g]$, (G_0, G_1) and (G_1, G_2) are $\mathbb{B} \times \mathbb{B}$-generic over N, $G_2 = G' \cap N$, and each filter contains p. This uses the fact that $\mathbb{B}$ is a *reasonable* forcing (in fact ccc). Letting x_i be the interpretation of τ by G_i for $i = 0, 1, 2$ we get $x_0 \simeq x_1$, $x_1 \simeq x_2$ and hence $x_0 \simeq x_2$. This is a contradiction as $x_0 \in V[g]$ and $x_2 = \tau_{G'}$. □

Using this fact, build a sequence of pairs of conditions (q_n, r_n) such that

$$(q_n, r_n) \Vdash_{\mathbb{B}\times\mathbb{B}} \tau_l < \tau_r$$

and such that $r_{n+1} \leq q_n$ for every $n < \omega$. Now let X be a countable elementary submodel of a sufficiently large $H(\theta)$ with transitive collapse $j : X \to H$. Let $\{g_n \mid n < \omega\}$ be a sequence of filters on $j(\mathbb{B})$ with the property that $g_{n+1} \times g_n$ generates an H-generic filter on $j(\mathbb{B}) \times j(\mathbb{B})$ and $r_n \in g_n$ for each $n < \omega$. Let $x_n = \tau_{g_n}$. Then $\{x_n \mid n < \omega\}$ is $<$-decreasing giving the the desired contradiction. □

The preceding remarks suggest that we add the existence of approximations to such an embedding to the induction hypothesis. This was how we came to formulate I_α in [29], the assertion that whenever $G \subset P(\omega_1)/NS$ is V-generic and $g \subset Col(\omega, \omega_1^V)$ is $V[G]$-generic, there exists

$$\bar{\pi} : J_{\bar{\alpha}}(\mathbb{R})^{V[g]} \to J_{\bar{\alpha}}(\mathbb{R})^{V[G][g]}$$

which is Σ_1 elementary and fixes ordinals. In 4.2 we will use W^*_α and I_α to get $I_{\alpha+\omega}$. If α is not in the range of π_{NS} this will give $W^*_{\alpha+\omega}$ easily. Otherwise the arguments of section 3 will produce the requires witnessing structures.

4.1. $L(\mathbb{R})$ **and Cohen Forcing.** Recall that the ordinal height of the transitive structure $J_\alpha(\mathbb{R})$ is $\omega\alpha$ and that the new sets of reals appearing in $J_{\alpha+1}(\mathbb{R})$ are precisely the new sets which are first order definable over $J_\alpha(\mathbb{R})$, that is

$$P(\mathbb{R}) \cap J_{\alpha+1}(\mathbb{R}) = P(\mathbb{R}) \cap \utilde{\Sigma}^1_\omega(J_\alpha(\mathbb{R})).$$

We say that α begins a gap if there is no $\beta < \alpha$ with $J_\beta(\mathbb{R})$ a Σ_1 elementary (with real parameters) submodel of $J_\alpha(\mathbb{R})$.

Lemma 24. *Suppose $g \subset Col(\omega, \kappa)$ is V-generic and α begins a gap in $L(\mathbb{R})^{V[g]}$. Suppose $J_\alpha(\mathbb{R})^{V[g]} \models AD$. Then there is an ordinal α_0 and a Σ_1 embedding $j : J_{\alpha_0}(\mathbb{R}) \to J_\alpha(\mathbb{R})^{V[g]}$.*

Proof. There are uniformly Σ_1 definable functions

$$f_\alpha : [\omega\alpha]^{<\omega} \times \mathbb{R} \to J_\alpha(\mathbb{R})$$

which are surjective. Using these we define a Σ_1 function F as follows. Given a real x, decode a sequence $(x_0, ..., x_n)$ of reals and a real y and suppose there is a finite sequence F such that

$$J_\alpha(\mathbb{R}) \models \phi_{y(0)}((x_0, ..., x_n), f_\alpha(F, \hat{y}))$$

where $\hat{y}(n) = y(n+1)$ and $(\phi_k \mid k < \omega)$ enumerates Σ_1 formulae with two free variables. Let F^* be the $<_{KB}$-least such F and set $F(x) = f_\alpha(F^*, \hat{y})$. F is a uniformly Σ_1 partial map and if α begins a gap then F as defined over $J_\alpha(\mathbb{R})$ is surjective. Let T be the tree of the scale on the universal Σ_1 set defined over $J_\alpha(\mathbb{R})$. T is in V by homogeneity. Let $M = F[\mathbb{R}^V]$ where F is computed in $J_\alpha(\mathbb{R})$ of $V[g]$. Using T we see that $\mathbb{R} \cap M = \mathbb{R}^V$. It follows that $M \simeq J_{\alpha_0}(\mathbb{R})^V$ for some ordinal α_0 and the inverse of the collapse is the desired map j. □

Lemma 25. *Suppose α is an inadmissible limit ordinal which begins a gap.*

1. *There is a surjective function $f : \mathbb{R} \to J_\alpha(\mathbb{R})$ which is Δ_1 definable over $J_\alpha(\mathbb{R})$ from a real z_0.*
2. *If $A \in P(\mathbb{R}) \cap J_{\alpha+1}(\mathbb{R})$ then A is projective in a set $D \in P(\mathbb{R})$ which is Δ_1 definable over $J_\alpha(\mathbb{R})$ from a real.*
3. *If $\underset{\sim}{\Delta}_{2k+1}(J_\alpha(\mathbb{R}))$-determinacy holds then the pointclasses*

$$\underset{\sim}{\Pi}_{2k+2}(J_\alpha(\mathbb{R})) \text{ and } \underset{\sim}{\Sigma}_{2k+3}(J_\alpha(\mathbb{R}))$$

   have the scale property.

Proof. Inadmissibility of $J_\alpha(\mathbb{R})$ together with a Skolem hull argument gives a map $g : \mathbb{R} \to \omega\alpha$ which is cofinal. Using the uniform Σ_1 Skolem function this can be turned into the desired map f. For (2) note that every such A can obtained from a Δ_1 set of the form

$$D = \{(x, x_1, ..., x_k) \in \mathbb{R}^{k+1} \mid J_\alpha(\mathbb{R}) \models \phi(x, f(x_1), ..., f(x_k), f(r))\}$$

by taking projections and complements, for some Σ_0 formula ϕ and real r. Part (3) follows from the second periodicity theorem. □

Lemma 26. *Suppose $J_\alpha(\mathbb{R}) \models \mathsf{AD}$ and α begins a gap. Suppose $g \subset Col(\omega, \omega)$ is generic over V. Then there exists a Σ_1 elementary embedding*

$$j : J_\alpha(\mathbb{R}) \to J_\alpha(\mathbb{R})^{V[g]}.$$

Furthermore, for any $k \geq 0$,

1. *if $\underset{\sim}{\Sigma}_1(J_\alpha(\mathbb{R})) \cap P(\mathbb{R})$ has the Baire property then j is Σ_2 elementary.*
2. *if α is inadmissible, $\underset{\sim}{\Delta}_{2k+1}(J_\alpha(\mathbb{R}))$ games are determined and $\underset{\sim}{\Sigma}_{2k+3}(J_\alpha(\mathbb{R}))$ sets have the Baire property then j is Σ_{2k+4} elementary.*
3. *if $J_\alpha(\mathbb{R})$ is admissible then j is fully elementary.*

Proof. We think of the reals as ω^ω. For $p \in \omega^{<\omega}$ let N_p denote the neighborhood determined by p. Let $\mathcal{B}$ denote the σ-algebra of sets of reals in $L(\mathbb{R})$ which have the Baire property. Let $\mathbb{B}$ denote the quotient algebra $\mathcal{B}/I$ where I is the ideal of meager sets. Clearly the map

$$\pi : Col(\omega, \omega) \to B$$

be defined by $\pi(p) = [N_p]$ is a dense embedding so if $g \subset Col(\omega, \omega)$ is V-generic then g induces an ultrafilter U_g on B. That is, a set A is in U_g if and only if $A \cap N_p$ is comeager in N_p for some $p \in g$ We first construct the Σ_1 embedding j. In $V[g]$ we form the ultrapower $Ult(J_\alpha(\mathbb{R}), U_g)$ using functions inside $J_\alpha(\mathbb{R})$. We may assume that these functions are total with domain $\mathbb{R}$. We first show that $Ult(J_\alpha(\mathbb{R}), U_g)$ is well-founded. Assume a condition p forces that $(\dot{f}_n)$ is a decreasing sequence in the ultrapower. For $s \in \omega^{<\omega}$ define a condition $p_s \in Col(\omega, \omega)$, a set A_s, and a function f_s all in $J_\alpha(\mathbb{R})$ such that

(a) $p \subseteq p_\emptyset$
(b) $\{p_{s^\frown n} \mid n \in \omega\}$ is a maximal antichain below p_s
(c) $A_s \subset N_s$ is comeager in N_s,
(d) $p_s \Vdash_{Col(\omega,\omega)} \dot{f}_{lh(s)} = f_s$.
(e) $s \subset t$,$s \neq t$, and $x \in A_t$ implies $f_t(x) \in f_s(x)$.

By the Baire Category Theorem

$$\bigcap_{n<\omega} \bigcup_{lh(s)=n} A_s \neq \emptyset$$

so there are $x, h \in \omega^\omega$ so that $x \in A_{h \restriction n}$ for every $n < \omega$ and hence $\{f_{h\restriction n}(x) \mid n < \omega\}$ is an $\in$-decreasing sequence, giving the desired contradiction. Thus the ultrapower has a transitivization M. For the tighter results (1) through (3) we use the uniformizations provided by the scale property to prove Los. However, we include the following uniformization result as it yields a more elementary proof in many cases that determinacy hypotheses are preserved by Cohen forcing.

Claim 27. Suppose $A \subset \mathbb{R} \times \mathbb{R}$ and $A \in J_\alpha(\mathbb{R})$. Then there is a Borel function f and a comeager set D such that if $x \in D$ and there is y such that $(x, y) \in A$ then $(x, f(x)) \in A$.

Proof. Fix such an A and consider the following game G. I and II alternate playing finite sequences of integers to produce a real x. In addition II plays finite sequences of integers at each turn to produce another real y. II wins the round if

$$\exists z\ (x, z) \in A \to (x, y) \in A.$$

This game is easily coded as an integer game in $J_\alpha(\mathbb{R})$ and hence it is determined. If II wins then the set of cooperative plays D is the desired comeager set. We must rule out the case that I wins. Suppose toward a contradiction that I wins via a strategy τ. Let $p \in \omega^{<\omega}$ be the first move of τ. We must have $N_p \cap p[A]$ comeager in N_p, otherwise II could play so that the cooperative real belongs to $\mathbb{R} \setminus p[A]$. Let $B \subset \mathbb{R} \times \mathbb{R}$ denote the set of all plays in G where I follows τ. Thus B and A are disjoint. Let f be a uniformizing function for the $\Pi^1_1(\tau)$ set $(\mathbb{R} \times \mathbb{R}) \setminus B$. By the Baire property, f is continuous on a comeager set D_0. Let $D = D_0 \cap p[A]$. Thus D is comeager in N_p. We can now design a play for II so that the cooperative play x belongs to D but II's side play is equal to $f(x)$, thus giving the desired contradiction. □

We get the following form of Los's Theorem. For functions $f_1, ..., f_n \in J_\alpha(\mathbb{R})$ and a Σ_0 formula ϕ,

$$Ult(J_\alpha(\mathbb{R}), U_g) \models \phi([f_1], ..., [f_n])$$

if and only if

$$\{r \in \mathbb{R} \mid J_\alpha(\mathbb{R}) \models \phi(f_1(r), ..., f_n(r))\} \in U_g.$$

For the existential step, we may assume that $\beta < \alpha$ begins a gap and $J_\beta(\mathbb{R}) \models \exists Y \phi(Y, f(x))$ for all x in some set $B \in U_g$. Let A denote the set of (x, y) such that $\phi(F(y), f(x))$ holds in $J_\beta(\mathbb{R})$ and let g^* uniformize this A as in the claim and define $g(x)$ to be $F(g^*(x))$ if defined and 0 otherwise. Thus

$$\{x \in \mathbb{R} \mid J_\alpha(\mathbb{R}) \models \phi(g(x), f(x))$$

also belongs to U_g so that $\phi([g], [f])$ holds in the ultrapower as desired. It follows that the ultrapower map $j : J_\alpha(\mathbb{R}) \to M$ is Σ_1 elementary since it is cofinal. There is a minor detail which we have ignored. The language of $L(\mathbb{R})$ includes a name for $\mathbb{R}$. Our name in the ultrapower is $[f_\mathbb{R}]$ which is the equivalence class of the function which only takes the value $\mathbb{R}$. It follows from Los that the ultrapower satisfies the sentence asserting that it is a level of $L(\mathbb{R})$. If σ is a standard term for a real then $f_\sigma(r) = \sigma_r$ defines a function which collapses to $\sigma_g \in M$. It follows that $M = J_\gamma(\mathbb{R})$ in $V[g]$. In $J_\alpha(\mathbb{R})$ any well-ordered union of meager sets is meager. It follows that for any ordinal η and function $f : \mathbb{R} \to \eta$ in $J_\alpha(\mathbb{R})$ there is a dense set $D \subset \omega^\omega$ such that f is constant on a comeager subset of N_p for any $p \in D$. It follows that $\gamma = \alpha$ and that $j \restriction \omega\alpha$ is the identity. For (1) we form the ultrapower using partial functions f which are $\undertilde{\Sigma}_1$-definable over $J_\alpha(\mathbb{R})$. Scales give the requisite uniformizations and the Baire property gives Los for Π_1-formulae. For (3) at some $k < \omega$ we use Σ_{2k+3}-definable partial functions whose domain includes a set in U_g. (3) follows as $J_{\alpha+1}(\mathbb{R}) \models AD$ in this case (see [11]). $\square$

We now present a sufficient condition for the pointclass $\Gamma = \utilde{\Sigma}_1(J_\alpha(\mathbb{R}))$ to have the Baire property. This will involve the C_Γ operation and related concepts. Of course this is automatic if α has countable cofinality. So we assume that α begins a gap, $J_\alpha(\mathbb{R}) \models \mathsf{AD}$, and $\mathrm{cof}(\alpha) > \omega$. For our purposes, we define

$$C_\Gamma(x) = \mathbb{R} \cap OD_x^{<\alpha},$$

that is, for reals x, y we put $y \in C_\Gamma(x)$ if there is $\beta < \alpha$ such that y is ordinal definable over $J_\beta(\mathbb{R})$ from the parameter x. Similarly

$$C_\Gamma(a) = P(a) \cap OD_{a \cup \{a\}}^{J_\alpha(\mathbb{R})}$$

for a countable transitive set a. A transitive set M is Γ-closed if $a \in M$ implies $C_\Gamma(a) \in M$. The model $L^\Gamma[x]$ is the minimal transitive model of height ω_1 which contains x and is Γ-closed. Finally, we say that ω_1 is Γ-inaccessible to reals if

$$\omega_1^{L^\Gamma[x]} < \omega_1$$

for every real x.

Lemma 28. *Assume $J_\alpha(\mathbb{R}) \models AD$, α begins a gap, cof(α) $> \omega$ and ω_1 is Γ-inaccessible to reals where $\Gamma = \utilde{\Sigma}_1(J_\alpha(\mathbb{R})) \cap P(\mathbb{R})$. Then*

1. *Γ has the Baire Property.*
2. *ω_1 is Γ-inacessible to reals in $V[g]$ whenever g is Cohen generic over V.*

Proof. Let A be a set in Γ. Let x be a real such that A is Σ_1-definable over $J_\alpha(\mathbb{R})$ from x. Let N be a rank initial segment of $L^\Gamma[x]$ containing its reals. There are comeager many Cohen generics over N so if each one lands in $\mathbb{R} \setminus A$ we're finished. Assume therefore that there is g_0 which is Cohen generic over N with $g_0 \in A$. Let $\beta < \alpha$ be least such that $J_\beta(\mathbb{R}) \models g_0 \in A$. Let $\beta_1 = \beta + \omega$ and let T be the tree of the scale on the Universal $\Sigma_1(J_{\beta_1}(\mathbb{R}))$ set. Note that N is a rank initial segment of $L[T, N]$. Now assume toward a contradiction that there exists a sequence of open dense sets $D_n \subseteq \mathbb{R}$ such that

$$g \in \bigcap_{n \in \omega} D_n \Rightarrow g \in (\neg A)^{J_\beta(\mathbb{R})}.$$

Then for any real z coding N, the model $J_{\beta_1}(\mathbb{R})$ satisfies the sentence asserting the existence of g_0, $\{D_n \mid n \in \omega\}$ and β satisfying

1. g_0 is Cohen generic over the model coded by z

(2) $g_0 \in A^{J_\beta(\mathbb{R})}$
(3) $g \in \bigcap_{n\in\omega} D_n \Rightarrow g \in (\neg A)^{J_\beta(\mathbb{R})}$

This is a Σ_1 sentence and so it holds in $L[T,z]$ by absoluteness. Thus for every g which is $L[T,z]$-generic, $g \in (\neg A)^{J_\beta(\mathbb{R})}$. Now, g_0 is generic over $L[T,N]$. Let G be generic over $L[T,N][g_0]$ for $Col(\omega,N)$. Then g_0 is generic over $L[T,N][G]$. But G codes N which is a contradiction. Thus by the Baire Property there is a condition p such that $g \in A^{J_\beta(\mathbb{R})}$ for comeager many g below p. Let O be the union of all neighborhoods which have this property. We claim that $A \setminus O$ is meager. Otherwise there is an $L^\Gamma[x,y]$ generic g which lands in $A \setminus O$ where y codes the opern set O. But $A \setminus O$ is Σ_1 so the preceeding analysis produces a q such that comeager many g below q land in $A \setminus O$, a contradiction as $N_q \subseteq O$. For part (2) note that we interpret Γ in $V[g]$ as $(\Sigma_1(J_\alpha(\mathbb{R})) \cap P(\mathbb{R}))^{V[g]}$. Let τ be a standard term for a real. It suffices to show that $C_\Gamma(\tau_g)$ is countable. Assume otherwise. Thus we may assume that there is a single Σ_1 formula $\psi(x,y,z)$ and a condition p which forces that $\{r_\eta \mid \eta < \omega_1\}$ is a distinct sequence of reals where

$$r_\eta = \{n < \omega \mid J_\alpha(\mathbb{R})^{V[g]} \models \psi(\eta, n, \tau_g)\}.$$

Let z be a real coding τ and let $\phi(\eta,q,n)$ be the Σ_1 formula which asserts that $q \le p$, z codes a term τ and

$$q \Vdash_{Col(\omega,\omega)} \psi(\eta,n,\tau).$$

Setting $r*_\eta = f[\{(q,n) \mid \phi(\eta,q,n)\}]$ where $f : Col(\omega,\omega)\times\omega \to \omega$ is a fixed bijection we see that each r^*_η belongs to $C_\Gamma(z)$ and that they are distinct, contradicting our hypotheses. □

Lemma 29. *Assume $J_\alpha(\mathbb{R})$ is inadmissible and models AD. Suppose that whenever $G \subset P(\omega_1)/NS$ is V-generic and $g \subset Col(\omega,\omega_1^V)$ is $V[G]$-generic that there exists a Σ_1 embedding*

$$\pi : J_{\pi_{NS}(\alpha)}(\mathbb{R})^{V[g]} \to J_{\pi_{NS}(\alpha)}(\mathbb{R})^{V[G][g]}$$

where π_{NS} is the ultrapower map derived from G. Then $J_{\pi_{NS}(\alpha)}(\mathbb{R})^{V[g]}$ is inadmissible and $\pi_{NS}(\alpha)$ is independent of the generic.

Proof. Let f be a total surjective map from $\mathbb{R}$ to $J_\alpha(\mathbb{R})$ which is Σ_1 definable over $J_\alpha(\mathbb{R})$, as in Lemma 24. Thus the same definition gives rise to a total map in $J_{\pi_{NS}(\alpha)}(\mathbb{R})^{V[G]}$. We claim that

$$\Gamma = \underset{\sim}{\Sigma}_1(J_\alpha(\mathbb{R})) \cap P(\mathbb{R})$$

has the Baire Property. We may assume that $\mathrm{cof}(\alpha) > \omega$. It suffices to show that ω_1 is Γ-inaccessible to reals. Suppose to the contrary that $x \in \mathbb{R}$ and $L^\Gamma[x]$ thinks that every set is countable. We have

$\pi_{NS}(L^\Gamma[x]) = L^{\pi_{NS}(\Gamma)}[x]$ so there is $r \in V[G]$ which is ordinal definable from x over some level $J_\beta(\mathbb{R})^V[G]$ which codes a well-ordering of length ω_1^V. It follows that r is similarly definable over $J_\beta(\mathbb{R})^{V[G][g]}$ and hence in $J_\beta(\mathbb{R})^{V[g]}$ using π. This is a contradiction as $V[g]$ is a homogeneous extension and we would therefore have $r \in V$. Thus Γ has the Baire property and so the map

$$j : J_{\pi_{NS}(\alpha)}(\mathbb{R}^{V[G]}) \to J_{\pi_{NS}(\alpha)}(\mathbb{R})^{V[G][g]}$$

is Σ_2 elementary by Lemmas 26 and 27. It follows that f defines a total map in $J_{\pi_{NS}(\alpha)}(\mathbb{R}^V[G][g])$. Using π we conclude that f is total in $J_{\pi_{NS}(\alpha)}(\mathbb{R}^V[g])$ as well. It follows that $\pi_{NS}(\alpha)$ begins a gap in $L(\mathbb{R})^{V[g]}$ and hence that it is independent of the generic. □

4.2. **The Induction Step in the Inadmissible Case.** We now handle the induction step in the case that α is an inadmissible limit ordinal which begins a gap in the $L(\mathbb{R})$ of $V[g]$ where $g \subset Col(\omega, \omega_1)$ is any V-generic. For the remainder of the section we shall assume W_α^* in any such $V[g]$ and we also assume I_α as well as our hypotheses $WRP_{(2)}(\omega_2)$ and NS saturated. Thus in particular whenever $G \subset P(\omega_1)/NS$ is V-generic and $g \subset Col(\omega, \omega_1^V)$ is $V[G]$-generic,

(1) there is a Σ_1 embedding

$$\pi : J_\alpha(\mathbb{R})^{V[g]} \to J_\alpha(\mathbb{R})^{V[G][g]}$$

which satisfies $\pi \restriction \omega\alpha = id$, and

(2) $J_\alpha(\mathbb{R})^{V[g]} \models AD$.

Note that at this point it is not obvious that $J_\alpha(\mathbb{R})^{V[G][g]}$ is inadmissible.

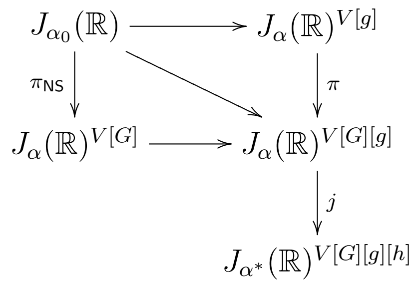
Lemma 30. *Let α_0 be least such that in some extension $\pi_{NS}(\alpha_0) \geq \alpha$. Then in every extension $\pi_{NS}(\alpha_0) \geq \alpha$.*

Proof. We first claim that α begins a gap in $V[G][g]$. Otherwise there is $\gamma < \bar{\alpha}$ which begins a gap I such that $\alpha \in I$. In $V[g]$ there is a real z and a Σ_1 formula ψ such that $J_\gamma(\mathbb{R}) \models \neg\psi(z)$ and $J_\alpha(\mathbb{R}) \models \psi(z)$. Thus the same would be true in $V[G][g]$, a contradiction. It now follows that α_0 begins a gap in $L(\mathbb{R})^V$. Otherwise in some $V[G]$ there is $\gamma < \bar{\alpha}$ which begins a gap I such that $\bar{\alpha} \in G$. This yields a similar contradiction. Note that $J_\alpha(\mathbb{R}) \models \mathsf{AD}$. Using Lemma 23 it is easy to see that there is a Σ_1 elementary embedding

$$j : J_{\alpha_0}(\mathbb{R}) \to J_\alpha(\mathbb{R})^{V[g]},$$

and by homogeneity there is such an embedding in any $V[g]$. It now follows that $\pi_{NS}(\alpha_0) \geq \alpha$ in every extension. □

The following diagram helps illustrate the steps of our proof.



Here $h \subset Col(\omega, \omega_2^V)$ is $V[G][g]$-generic and we may assume that the model $V[G][g][h]$ could be reorganized as $V[\bar{h}]$ for another $Col(\omega, \omega_2^V)$-generic. Of course, π_{NS} is the NS-generic ultrapower and $J_\alpha(\mathbb{R})^{V[G]}$ and $J_\alpha(\mathbb{R})^{V[G][g]}$ are connected via the ultrapower of Lemma 25. Note that $\pi_{NS}[\alpha_0] \leq \alpha \leq \pi_{NS}(\alpha)$ and it is possible that α_0 is a discontinuity point so that

$$\pi_{NS} : J_{\alpha_0}(\mathbb{R}) \to J_{\alpha_0}(\mathbb{R})^{V[G]}$$

is in general only Σ_1 elementary. The map π is given by I_α.

Step 1. We will first identify a level α^* of the $L(\mathbb{R})$ of $V^{Col(\omega,\omega_2)}$ in which AD holds. The construction of this level will allow us to show that a function f as in Lemma 24 witnessing the inadmissibility of α in $V[g]$ remains total in $J_{\alpha^*}(\mathbb{R})^{V[G][g][h]}$ and that the map π is Σ_2 elementary. Thus α is inadmissible in $V[G][g]$ as witnessed by the same function. Since $V[g][G][h]$ is a homogeneous extension of $V[g][G]$ we will get a map j as in Lemma 23.

Step 2. If $\pi_{NS}(\alpha_0) = \alpha$ then we will use Lemma 18 (with the appropriate mouse operator) to get more determinacy in V and hence in $V[G][g]$. A key point is that the real parameter used to define f as in Step 1 will be in V in this case. Otherwise $\pi_{NS}(\alpha_0) > \alpha$ and this extra determinacy comes for free.

Step 3. An argument similar to that of Lemma 21 then extends π to establish $I_{\alpha+\omega}$ from which $W^*_{\alpha+\omega}$ will follow easily.

We now identify the level α^* of the $L(\mathbb{R})$ of the universe after the collapse of ω_2 which is analogous to α. Recall that F denotes the uniform Σ_1 function from Lemma 23. We use $\forall^* \sigma$ to abbreviate "for a club of $\sigma \in [\omega_2]^\omega$".

Lemma 31. *There is function h and a term for an ordinal α^* such that for any $p \in Col(\omega, \omega_2)$, standard term for a real τ, and Σ_1 formula ϕ the following are equivalent.*

(1) $p \Vdash_{Col(\omega,\omega_2)} J_{\alpha^*}(\mathbb{R}) \models \phi(F(\tau))$
(2) $\forall^* \sigma\ p^\sigma \Vdash_{Col(\omega,otp(\sigma))} J_{h(\sigma\cap\omega_1)}(\mathbb{R}) \models \phi(F(\tau^\sigma))$

Proof. Using NS saturated we find a function $h : \omega_1 \to \alpha$ in V which always represents $\bar{\alpha}$ in the ultrapower. Let $\{S_\xi, h_\xi \mid \xi < \omega_1\}$ be such that $\{S_\xi \mid \xi < \omega_1\}$ is a maximal antichain in $P(\omega_1)/NS$ with S_ξ forcing $h_\xi : \omega_1 \to \alpha$ to represent $\bar{\alpha}$ in the generic ultrapower. Define $h(\eta) = h_\xi(\eta)$ where ξ is least such that $\eta \in S_\xi$. Thus h is the desired representative. We will use h together with the uniform Σ_1 function F to define an interpretation of the $J_{\bar{\alpha}}(\mathbb{R})$ of $V^{Col(\omega,\omega_1)}$ in $V^{Col(\omega,\omega_2)}$. Let

$$\Gamma = \Sigma_1(J_{\bar{\alpha}}(\mathbb{R})) \cap P(\mathbb{R}).$$

We claim that Γ has the Baire property in $V[G]$. If $\bar{\alpha} < \pi_{NS}(\alpha)$ then in fact Γ sets are determined in $V[G]$. Otherwise there is some extension $V[G]$ in which $\bar{\alpha} = \pi_{NS}(\alpha)$. In this case the argument of Lemma 11 applies to show that ω_1 is Γ-inaccessible to reals and thus we get the Baire property by Lemma 9. Thus $\Sigma_1(J_{h(\xi)}(\mathbb{R})) \cap P(\mathbb{R})$ has the Baire property for every $\xi < \omega_1$. We define a name for a structure

$$\dot{\mathcal{M}} = (\dot{M}, \dot{\in}_{\mathcal{M}}, \dot{=}_{\mathcal{M}})$$

as follows. For $p \in Col(\omega, \omega_2)$ and $\tau, \bar{\tau}$ standard terms for reals

$(p, \tau) \in \dot{M}$ if and only if

$$\forall^* \sigma\ p^\sigma \Vdash_{Col(\omega,otp(\sigma))} J_{h(\sigma\cap\omega_1)}(\mathbb{R}) \models F(\tau^\sigma) \text{ exists}$$

$(p, (\tau, \bar{\tau})) \in \dot{\in}_{\mathcal{M}}$ if and only if

$$\forall^* \sigma\ p^\sigma \Vdash_{Col(\omega,otp(\sigma))} J_{h(\sigma\cap\omega_1)}(\mathbb{R}) \models F(\tau^\sigma) \in F(\bar{\tau}^\sigma)$$

$(p, (\tau, \bar{\tau})) \in \dot{=}_{\mathcal{M}}$ if and only if

$$\forall^* \sigma\ p^\sigma \Vdash_{Col(\omega,otp(\sigma))} J_{h(\sigma\cap\omega_1)}(\mathbb{R}) \models F(\tau^\sigma) = F(\bar{\tau}^\sigma)$$

There is an abuse of notation here. By $(\tau, \bar{\tau})$ above we really mean the term for the ordered pair. We prove the following Los-type assertion.

Claim 32. For a condition p, terms $\tau_1, ..., \tau_k$, and a Σ_1 formula ϕ the following are equivalent.

(1) $p \Vdash_{Col(\omega,\omega_2)} \tau_1, ..., \tau_k \in \dot{M} \wedge \dot{\mathcal{M}} \models \phi(\tau_1, ..., \tau_k)$
(2) $\forall^* \sigma\ p^\sigma \Vdash_{Col(\omega,otp(\sigma))} J_{h(\sigma\cap\omega_1)}(\mathbb{R}) \models \phi(F(\tau_1^\sigma), ..., F(\tau_k^\sigma))$.

Proof. Note that $p \Vdash_{Col(\omega,\omega_2)} \tau \in \dot{M}$ if and only if $(p, \tau) \in \dot{M}$. The atomic cases follow easily. We handle negation as follows. Assume $\phi(\tau_1, ..., \tau_k)$ is $\neg\psi(\tau_1, ..., \tau_k)$, ψ is Σ_1, and the equivalence above holds for

$\psi(\tau_1, ..., \tau_k)$. Assume first that $p \Vdash_{Col(\omega,\omega_2)} \dot{\mathcal{M}} \models \phi(\tau_1, ..., \tau_k)$. Assume toward a contradiction that there is a stationary set $A_0 \subset [\omega_2]^\omega$ such that for $\sigma \in A$ we have

$$\neg p^\sigma \Vdash_{Col(\omega,otp(\sigma))} J_{h(\sigma\cap\omega_1)}(\mathbb{R}) \models \phi(F(\tau_1^\sigma), ..., F(\tau_k^\sigma)).$$

Thus by refining p^σ and pressing down we find a q below p and a stationary subset $A \subset A_0$ such that $\sigma \in A$ implies

$$q^\sigma \Vdash_{Col(\omega,otp(\sigma))} J_{h(\sigma\cap\omega_1)}(\mathbb{R}) \models \psi(F(\tau_1^\sigma), ..., F(\tau_k^\sigma)).$$

Similarly we get a stationary set B and a condition r below q such that for $\sigma \in B$ we have

$$r^\sigma \Vdash_{Col(\omega,otp(\sigma))} J_{h(\sigma\cap\omega_1)}(\mathbb{R}) \models \phi(F(\tau_1^\sigma), ..., F(\tau_k^\sigma)).$$

We now find an ordinal $\gamma < \omega_2$ above r so that $A \cap [\gamma]^\omega$ and $B \cap [\gamma]^\omega$ are both stationary in $[\gamma]^\omega$. Let $(\sigma_\xi \mid \xi < \omega_1)$ be a continuous, exhaustive chain in $[\gamma]^\omega$ and let $\bar{A} = \{\xi \mid \sigma_\xi \in A\}$ and similarly define $\bar{B}$. Let $G_A, G_B \subset P(\omega_1)/NS$ be V-generic with $\bar{A} \in G_A$ and $\bar{B} \in G_B$. Let $g \subset Col(\omega, \gamma)$ be generic over both $V[G_A]$ and $V[G_B]$ with $r \in g$. Thus

$$J_{\bar{\alpha}}(\mathbb{R})^{V[G_A][g]} \models \psi(F((\tau_1 \restriction \gamma)_g), ..., F((\tau_k \restriction \gamma)_g))$$

and

$$J_{\bar{\alpha}}(\mathbb{R})^{V[G_B][g]} \models \phi(F((\tau_1 \restriction \gamma)_g), ..., F((\tau_k \restriction \gamma)_g)).$$

By hypothesis there are Σ_1 embeddings

$$\bar{\pi}_A : J_{\bar{\alpha}}(\mathbb{R})^{V[g]} \to J_{\bar{\alpha}}(\mathbb{R})^{V[G_A][g]}$$

and

$$\bar{\pi}_B : J_{\bar{\alpha}}(\mathbb{R})^{V[g]} \to J_{\bar{\alpha}}(\mathbb{R})^{V[G_B][g]}.$$

Thus $J_{\bar{\alpha}}(\mathbb{R})^{V[g]}$ satisfies

$$\psi(F((\tau_1 \restriction \gamma)_g), ..., F((\tau_k \restriction \gamma)_g)) \wedge \neg\psi(F((\tau_1 \restriction \gamma)_g), ..., F((\tau_k \restriction \gamma)_g))$$

which is the desired contradiction. The other direction of the negation case follows similarly. We now treat the unbounded existential case. For the nontrivial direction suppose for a club of $\sigma \in [\omega_2]^\omega$ that

$$p^\sigma \Vdash_{Col(\omega,otp(\sigma))} J_{h(\sigma\cap\omega_1)}(\mathbb{R}) \models \exists x\ \phi(x, F(\tau^\sigma)).$$

For simplicity we assume there is only one parameter. For a real z, the set

$$\{x \mid \phi(F(x), F(z))\},$$

as interpreted in a level of $L(\mathbb{R})$ beginning a gap, is the projection of the tree of the Σ_1-scale on this set. We let $lw(z)$ denote witness obtained from the leftmost branch of this tree. A key point is that there is a Σ_1 formula ψ so that $\psi(u, z)$ holds if and only if $u = lw(z)$. We define a term $lw(\tau)$ as follows. For a condition q and a pair $n, m \in \omega$ we put the term $(q, (n, m))$ (abusing notation) in $lw(\tau)$ if and only if for a club of σ,

$$q^\sigma \Vdash_{Col(\omega, otp(\sigma))} J_{h(\sigma\cap\omega_1)}(\mathbb{R}) \models lw(\tau^\sigma)(n) = m.$$

We need to see that for a club of $\sigma \in [\omega_2]^\omega$

$$p^\sigma \Vdash_{Col(\omega, otp(\sigma))} J_{h(\sigma\cap\omega_1)}(\mathbb{R}) \models lw(\tau)^\sigma = lw(\tau^\sigma).$$

We will then have

$$p^\sigma \Vdash_{Col(\omega, otp(\sigma))} J_{h(\sigma\cap\omega_1)}(\mathbb{R}) \models \phi(F((lw(\tau)^\sigma), F(\tau^\sigma))$$

for each such σ as desired. Assume otherwise. We extract a condition q below p, integers n, m_1, m_2, an ordinal $\gamma < \omega_2$ above q and stationary sets $A, B \subset [\gamma]^\omega$ such that for $\sigma \in A$ we have

$$q^\sigma \Vdash_{Col(\omega, otp(\sigma))} J_{h(\sigma\cap\omega_1)}(\mathbb{R}) \models lw(\tau^\sigma)(n) = m_1$$

and for $\sigma \in B$ we have

$$q^\sigma \Vdash_{Col(\omega, otp(\sigma))} J_{h(\sigma\cap\omega_1)}(\mathbb{R}) \models lw(\tau^\sigma)(n) = m_2.$$

As in the negation case we get generics $G_A, G_B \subset P(\omega_1)/NS$ and $g \subset Col(\omega, \gamma)$ such that

$$J_{\bar{\alpha}}(\mathbb{R})^{V[G_A][g]} \models lw((\tau \restriction \gamma)_g)(n) = m_1$$

and

$$J_{\bar{\alpha}}(\mathbb{R})^{V[G_B][g]} \models lw((\tau \restriction \gamma)_g)(n) = m_2.$$

Using $\bar{\pi}$ we get a contradiction. This completes the proof of the claim. □

Now let $g \subset Col(\omega, \omega_2)$ be V-generic. In $V[g]$ we form a structure

$$(M, E) = (\dot{M}_g/\dot{=}_g, \dot{\in}_g/\dot{=}_g).$$

An easy argument which uses the fact that the club filter on $P([\omega_2]^\omega)$ is countably complete shows that M is well founded and so there is an isomorphism

$$j : (M, E) \to (N, \in)$$

for some transitive set N. The construction of the function F we have used is such that for any real r there is a real $\hat{r}$ obtained from r so that $F(\hat{r}) = r$ and $r, \hat{r}$ are Turing equivalent. Thus for any standard term for a real τ, there is a term $\hat{\tau}$ so that

$$\hat{\tau}_g = \hat{\tau}_g.$$

It follows that $\mathbb{R}^N = \mathbb{R}^{V[g]}$. By claim 31, N satisfies the sentence asserting that it is a level of $L(\mathbb{R})$ and thus $N = J_{\alpha^*}(\mathbb{R})$ in $V[g]$ for some ordinal α^* (which would seem to depend on g). We finish the proof of the lemma by showing that the following are equivalent for a Σ_1 formula ϕ, a condition $p \in Col(\omega, \omega_2)$ and a standard term for a real τ.

(1) $p \Vdash \dot{\mathcal{M}} \models \phi(\tau)$
(2) $p \Vdash J_{\alpha^*}(\mathbb{R}) \models \phi(F(\tau))$

Note that $\dot{M} \models F(\hat{\tau}) = \tau$ so $J^*_\alpha(\mathbb{R}) \models F(j(\hat{\tau}) = j(\tau)$. Since $j(\hat{\tau}) = \tau$ we see that $j = F$ which gives the equivalence above. □

Lemma 33. *The map π is Σ_2 elementary.*

Proof. Suppose that

$$J_{\bar{\alpha}}(\mathbb{R})^{V[g_1]} \models \forall x\ \exists y\ \phi(x, y, \tau^*_{g_1}),$$

where τ^* is a standard $Col(\omega, \omega_1)$ term for a real. We first redesign the term τ^*. Noting that $Col(\omega, \omega_2) \times Col(\omega, \omega_1)$ and $Col(\omega_2)$ have isomorphic completions, we can design a $Col(\omega, \omega_2)$ term τ so that whenever $\bar{g} \subset Col(\omega, \omega_2)$ is V-generic and gives rise to $g \subset Col(\omega, \omega_2)$ and $h \subset Col(\omega, \omega_1)$ with $V[\bar{g}] = V[g][h]$ we have $\tau_{\bar{g}} = \tau^*_h$. Let $\hat{\tau}$ be the canonical term obtained from τ so that $F(\hat{\tau}) = \tau$ in any extension. We claim that whenever $g \subset Col(\omega, \omega_2)$ is V generic

$$J_{\alpha_2}(\mathbb{R}) \models \forall y \in \mathbb{R}\ \exists x\ \phi(x, y, F(\hat{\tau})).$$

Assume otherwise. Thus there is a term for a real $\hat{\rho}$ and a condition $p \in Col(\omega, \omega_2)$ such that

$$p \Vdash_{Col(\omega,\omega_2)} J_{\alpha_2}(\mathbb{R}) \models \neg\exists x\ \phi(x, F(\hat{\rho}), F(\hat{\tau})).$$

By Lemma 25 we have a club of $\sigma \in [\omega_2]^\omega$ such that

$$p^\sigma \Vdash_{Col(\omega,otp(\sigma))} J_{h(\sigma\cap\omega_1)}(\mathbb{R}) \models \neg\exists x\ \phi(x, F(\hat{\rho}^\sigma), F(\hat{\tau}^\sigma)).$$

Fix γ above p such that there is such a club in $[\gamma]^\omega$. Let $G \subset P(\omega_1)/NS$ be V-generic and $\bar{g} \subset Col(\omega, \gamma)$ be V-generic below p. Then

$$V[G][\bar{g}] \models J_{\bar{\alpha}}(\mathbb{R}) \models \neg\exists x\ \phi(x, F((\hat{\rho} \restriction \gamma)_{\bar{g}}), F((\hat{\tau} \restriction \gamma)_{\bar{g}})).$$

Let $g \subset Col(\omega, \gamma)$ and $h \subset Col(\omega, \omega_1^V)$ be generics so that $V[\bar{g}] = V[g][h]$ and $V[G][\bar{g}] = V[G][g][h]$. Let $r = F((\hat{\rho} \restriction \gamma)_{\bar{g}})$ and note that $F((\hat{\tau} \restriction \gamma)_{\bar{g}}) = \tau^*_h$. Thus

$$V[G][g][h] \models J_{\bar{\alpha}}(\mathbb{R}) \models \neg\exists x\ \phi(x, r, \tau^*_h)$$

so that

$$V[h][g] \models J_{\bar{\alpha}}(\mathbb{R}) \models \neg\exists x\ \phi(x, r, \tau^*_h).$$

We know that

$$V[h] \models J_{\bar{\alpha}}(\mathbb{R}) \models \forall y \in \mathbb{R}\ \exists x\ \phi(x, y, \tau_h^*)$$

so we will get a contradiction if we can show that the embedding

$$i : J_{\bar{\alpha}}(\mathbb{R})^{V[h]} \to J_{\bar{\alpha}}(\mathbb{R})^{V[h][g]}$$

which exists by Lemma 8 is Σ_2 elementary. For this it suffices to see that Σ_1 over $J_{\bar{\alpha}}(\mathbb{R})^{V[h]}$ has the Baire property. This follows from Lemmas 10 and 11. Now fix a term τ^* such that from τ_g^* a total map $f : \mathbb{R} \to J_{\bar{\alpha}}(\mathbb{R})$ is Σ_1 definable over $J_{\bar{\alpha}}(\mathbb{R})$ in $V[g]$. Let $\phi(-, -, \tau^*)$ be the formula defining f. We have shown that whenever $\bar{g}$ is $Col(\omega, \omega_2)$ generic and gives rise to g, h as above then

$$J_{\alpha_2}(\mathbb{R})^{V[\bar{g}]} = J_{\alpha_2}(\mathbb{R})^{V[g][h]} \models \phi(-, -, \tau_h^*) \text{ defines a total map.}$$

Thus, whenever $g_2 \subset Col(\omega, \omega_2)$, $g_1 \subset Col(\omega, \omega_1)$ and $G \subset P(\omega_1)/NS$ are suitably generic then $V[g_2][g_1][G]$ thinks that $\phi(-, -, \tau_{g_1}^*)$ defines a total map over $J_{\alpha_2}(\mathbb{R})$. We claim that there is an embedding $\pi_2 : J_{\bar{\alpha}}(\mathbb{R})^{V[G][g_1]} \to J_{\alpha_2}(\mathbb{R})^{V[G][g_1][g_2]}$ which is Σ_1 elementary. To see this let let T be the tree of the scale on the universal Σ_1 set over the structure $J_{\alpha_2}(\mathbb{R})^{V[G][g_1][g_2]}$. Then $T \in V[G][g_1]$ by homogeneity. Using T we get a Σ_1 embedding

$$\pi_2 : J_{\gamma}(\mathbb{R})^{V[G][g_1]} \to J_{\alpha_2}(\mathbb{R})^{V[G][g_1][g_2]}$$

for some ordinal γ. Thus f is total at this level and it follows that $\gamma \geq \bar{\alpha}$. On the other hand, $\bar{\alpha}$ is the length of a ω_1-Universally Baire prewellordering from the perspective of $V[g_1]$ so that its length cannot change in passing to $V[g_1][G]$, a ccc extension under our hypotheses. Thus $\gamma = \bar{\alpha}$. Using π_2 we see that f is total in $J_{\bar{\alpha}}(\mathbb{R})^{V[G][g_1]}$ as desired. It follows that π is Σ_2-elementary. Note that α_2 is now independent of the generic. □

Lemma 34. $J_{\alpha+\omega}(\mathbb{R})^{V[G][g]} \models AD$

Proof. Let α_0 be as in Lemma 30. If $\pi_{NS}(\alpha_0) > \alpha$ in every extension than this holds automatically. Otherwise in some $V[G]$ we have $\pi_{NS}(\alpha_0) = \alpha$. We have just shown that α is inadmissible in $V[G][g]$. Since α begins a gap in $V[G]$ it must be the case that α is inadmissible in $V[G]$ as well. Thus α is inadmissible in V. Let z_0 be a real from which a surjective total map from $\mathbb{R}$ to $J_\alpha(\mathbb{R})$ is Σ_1-definable. It follows that f remains total in $J_\alpha(\mathbb{R})$ of $V[G][g]$ and $V[g]$ (this use the Baire property). We conclude now that $\pi_{NS}(\alpha_0) = \alpha$ in all extensions. We now perform a PD induction as in Lemma 18 relative to the mouse operator $\mathcal{M}$ from [19]. Note that $W_{\alpha_0}^*$ holds in V. To better correspond with the notation of [19] let us be more explicit about our inadmissibility witness. Fix a formula $\phi(u,v)$ which is Σ_1 in the language of $L(\mathbb{R})$ and so that the map which sends a real x to the least β which so that $J_\beta(\mathbb{R}) \models \phi(x, z_0)$ is total and cofinal. The operation $\mathcal{M}$ associates to a countable transitive set a which codes z_0 the least level of $Lp(a)$ which satisfies a certain sentence ψ which we now describe. If $\mathcal{M}(a)$ is any a-mouse and $g \subset Col(\omega, a)$ is generic then there is a real $z(g,a)$ which is interconstructible with the pair (g,a) so that $\mathcal{M}(a)[g]$ is a z-mouse. Let $\sigma \in \mathcal{M}(a)$ be term so that whenever g is such a generic

$$\{(\sigma_g)_i \mid i > 0\} = \{\rho_g \mid \rho \in L_1(a)\}$$

and $(\sigma_g)_0 = z_0$. We let the sentence ψ assert that whenever g is generic there is a γ so that $\mathcal{M}(z(g,a)) \restriction \gamma$ is a (ϕ_n^*, σ_g)-witness where $\phi_n^*(v)$ asserts that there is an ordinal α for which $\omega\alpha + n$ exists and

$$J_\alpha(\mathbb{R}) \models \forall i > 0\ \phi(v_i, v_0).$$

As in [19] the mouse operator $\mathcal{M}$ just described relativizes well. It also determined itself on generic extensions. Thus by Lemma 18, $H(\omega_3)$ is closed under $M_n^{\mathcal{M},\#}$ for every $n < \omega$. This gives $W_{\alpha+1}^*$ by 1.38 of [?] By repeating this induction ω times we get $W_{\alpha+\omega}^*$ as desired. □

Lemma 35. *$I_{\alpha+\omega}$ holds.*

Proof. We will let $\underset{\sim}{\Delta}_n$ denote the pointclass $\underset{\sim}{\Delta}_n(J_\gamma(\mathbb{R}))$ where γ indexes the least level where the map f is total in whatever universe is under consideration, and similarly for $\underset{\sim}{\Sigma}_n$ and $\underset{\sim}{\Pi}_n$. By Σ_n-Los in clauses (3) and (6) below we mean that Claim 31 holds for Σ_n formulae. Let $g_i \subset Col(\omega, \omega_i)$ for $i = 1, 2$ and $G \subset P(\omega_1)/NS$ be appropriately generic. We will show by induction on $k \geq 0$ that

(1) π is Σ_{2k+2}
(2) $\underset{\sim}{\Delta}_{2k+1}$-determinacy holds in $V[g_1]$
(3) Σ_{2k+2}-Los holds
(4) $\underset{\sim}{\Delta}_{2k+1}$-determinacy holds in $V[g_2]$
(5) $\bar{\pi}$ is Σ_{2k+3} elementary
(6) Σ_{2k+3}-Los holds
(7) $\underset{\sim}{\Sigma}_{2k+3}$ sets have the Baire Property in $V[g_1]$.

We will handle the base case $k = 0$ and it will be clear how the other cases go. We will assume that we are in the harder scenario, namely that $\pi_{NS}(\alpha_0) > \alpha$ in every extension so we are stuck with the parameter τ^*. We have already shown that π is Σ_2-elementary so (1) is established. Now fix a set $A \in \underset{\sim}{\Delta}_1$ of $V[g_1]$. A remains $\underset{\sim}{\Delta}_1$ in $V[G][g_1]$ so that third periodicity argument of Lemma 21 applies giving (2). We work toward

(3) and (4). Let ρ be a standard $Col(\omega,\omega_2)$ term for a real and let ϕ,ψ be Σ_1 formula so that $\phi(-,\rho)$ and $\psi(-,\rho)$ define a set in $\underset{\sim}{\Delta}_1$ in $V^{Col(\omega,\omega_2)}$. We need to show that for a club of σ

$$\emptyset \Vdash_{Col(\omega,otp(\sigma))} J_{h(\sigma\cap\omega_1)}(\mathbb{R}) \models \forall x\ \phi(x,\rho^\sigma) \vee \psi(x,\rho^\sigma)$$

in order to implement the $\mathrm{WRP}_{(2)}(\omega_2)$ argument. This is an instance of (3) which we now establish. Modulo the step for negation (which follows as in Claim 31) we only need to prove the existential step for Σ_2-formulae. So suppose more generally that for a club of σ

$$p^\sigma \Vdash_{Col(\omega,otp(\sigma)} J_{h(\omega_1\cap\sigma)}(\mathbb{R}) \models \exists x\psi(x,\tau^\sigma)$$

where ψ is Π_1. Since $\underset{\sim}{\Pi}_2$ over $J_{h(\omega_1\cap\sigma)}(\mathbb{R})$ has the scale property we can let $w(\tau^\sigma)$ be the standard $Col(\omega,otp(\sigma)$ term for the leftmost witness. Thus

$$p^\sigma \Vdash_{Col(\omega,otp(\sigma)} J_{h(\omega_1\cap\sigma)}(\mathbb{R}) \models \psi(w(\tau^\sigma),\tau^\sigma)$$

for a club of $\sigma\in[\omega_2]^\omega$ We define $w(\tau)$ as the set of $(q,(n,m))$ so that $q\le p$ and

$$q^\sigma \Vdash_{Col(\omega,otp(\sigma)} w(\tau^\sigma)(n)=m.$$

We will have $w(\tau)^\sigma = w(\tau^\sigma)$ for a club of σ provided we can rule out the following the existence of r,n,m_1,m_2 with $m_1<m_2$ and stationary sets $S,T\subset[\omega_2]^\omega$ such that $\sigma\in S$ implies

$$r^\sigma \Vdash_{Col(\omega,otp(\sigma)} w(\tau^\sigma)(n)=m_1,$$

$\sigma\in T$ implies

$$r^\sigma \Vdash_{Col(\omega,otp(\sigma)} w(\tau^\sigma)(n)=m_2.$$

Reflecting S,T to some $\gamma<\omega_2$ we can find generics $G_S,G_T\subset P(\omega_1)/NS$ and $g\subset Col(\omega,\gamma)$ as in Claim 31 so that

$$V[G_S][g]\models w((\tau\restriction\gamma)_g)(n)=m_1$$

and

$$V[G_T][g]\models w((\tau\restriction\gamma)_g)(n)=m_2.$$

Note that $\underset{\sim}{\Pi}_2$ is scaled in $V[g]$. The key point is that π being Σ_2 is enough to yield a contradiction as in Claim 31. This is because the Π_1 set

$$\{x \mid J_{\bar\alpha}(\mathbb{R})^{V[g]}\models\psi(x,(\tau\restriction\gamma)_g)\}$$

has a scale whose norms are $\underset{\sim}{\Delta}_2$. This gives (3). (4) may now be finished via a $\mathrm{WRP}_{(2)}(\omega_2)$ argument as in Lemma 21. Condition (5) now follows from the scale property in $V[g_2]$ using the map π_2 as in Lemma 32. (6) now follows as in the existential step. We get (7) because the argument of (2) now shows that $\underset{\sim}{\Pi}_2$ determinacy holds in $V[g_1]$. This gives $\underset{\sim}{\Sigma}_3$ has the Baire property in $V[g_1]$ be standard arguments (see [10]). With

Lemma 26 in mind we may now argue that $\bar{\pi}$ is Σ_4 as in Lemma 33, and continue the induction. □

Lemma 36. *$W^*_{\alpha+\omega}$ holds in $V[g]$ whenever $g \subset Col(\omega, \omega_1)$ is V-generic.*

Proof. If $\pi_{NS}(\alpha_0) = \alpha$ then we are done by Lemma 34. Otherwise we have always have $\pi_{NS}(\alpha_0) > \alpha + \omega$. We get $W^*_{\alpha_0}$ in V and hence $W^*_{\pi_{NS}(\alpha_0)}$ in $V[G][g]$. Using π we get $W^*_{\alpha+\omega}$ in $V[g]$ as desired. The main issue here is $\omega_1 + 1$ iterability. This is addressed by the map into $J_{\alpha_2}(\mathbb{R})$ of $V[g][g_2]$. □

At this point we can conclude that $J_{\kappa_{\mathbb{R}}+1}(\mathbb{R})$ satisfies AD in $V[g]$ and hence in V, where $\kappa_{\mathbb{R}}$ indexes the least admissible level of the $L(\mathbb{R})$ hierarchy. This hypothesis is known as Inductive Determinacy.

5. The Admissible Cases

Let us fix $g \subset Col(\omega, \omega_1)$ which is V-generic, and a critical ordinal γ in $V[g]$ of type (3). That is, letting α be the strict sup of the critical ordinals $< \gamma$, we have $\alpha < \gamma$. We assume that W^*_α holds in $V[g]$, and we wish to show that $W^*_{\gamma+1}$ holds in $V[g]$. The analysis of scales in $L(\mathbb{R})$ shows that α begins a Σ_1 gap $[\alpha, \beta]$, and $J_\alpha(\mathbb{R})$ is admissible. The possibilities are that $\alpha = \beta = \gamma - 1$ (the admissible empty gap), that $\alpha < \beta = \gamma - 1$ (the strong gap), or that $\alpha < \beta = \gamma$ (the weak gap). But for the most part, we do not need to distinguish these three cases here. We also assume $WRP_{(2)}(\omega_2)$, and that NS is saturated. Our overall plan is:

Step 1. Working in $V[g]$, we construct a mouse N and iteration strategy Σ^g which code up truth at the end of the gap $[\alpha, \beta]$. N will be a mouse over some real parameter z.

Step 2. Letting $\tau_g = z$, we show that N and Σ^g yield a mouse N_τ over τ and ω_2-iteration strategy Σ for N_τ, both in V, via the equations $N_\tau[g] = N$, and $\Sigma = \Sigma^g \restriction V$.

Step 3. We show that Σ extends to act on $H(\omega_3)$.

Step 4. We then further extend Σ so that it acts on all trees in the $H(\omega_1)$ of $V[g][h]$, where $h \subset Col(\omega, \omega_2^V)$ is $V[g]$-generic. At the same time we find versions $[\alpha^H, \beta^H]$ of our gap $[\alpha, \beta]$ in $V[g][H]$, whenever $H \in V[g][h]$, along with appropriately elementary embeddings from $J_\beta(\mathbb{R})^{V[g]}$ to $J_{\beta^H}(\mathbb{R})^{V[g][H]}$.

Step 5. We proceed as in the inadmissible case, but using Σ-mice with Woodin cardinals to witness $W^*_{\gamma+1}$ in $V[g]$. Core model theory gives the desired Σ-mice in the NS-generic ultrapower, and an absoluteness argument using the resemblance established in step 4 implies that these mice are in $V[g]$.

Steps 1 and 2 follow [19] closely. The only difference here is that we want N to have ω Woodin cardinals, so that we can lift the gap $[\alpha, \beta]$ to $V[g][h]$, for h generic over $Col(\omega, \omega_2)$, using an $\mathbb{R}$-genericity iteration over $V[g][h]$. We now proceed to details.

Definition 37. Let Γ be the pointclass $\Sigma_1^{J_\alpha(\mathbb{R})}$. What is called the envelope of Γ, or ENV(Γ), is the class of all $A \subseteq \mathbb{R}$ which are countably captured by Γ in that there is a real x such that for any countable $\sigma \subseteq \mathbb{R}$, $A \cap \sigma$ is $OD^{<\alpha}(\sigma, x)$. The analysis of scales from [ScLR] shows that if $\alpha = \beta$ or $[\alpha, \beta]$ is strong, then

$$\text{ENV}(\Gamma) = J_{\beta+1}(\mathbb{R}) \cap P(\mathbb{R}),$$

and if $[\alpha, \beta]$ is weak, then

$$\text{ENV}(\Gamma) = J_{\beta}(\mathbb{R}) \cap P(\mathbb{R}).$$

If $\alpha = \beta$ or $[\alpha, \beta]$ is strong, put

$$e(\Gamma) = \{A \subseteq \mathbb{R} \mid A \text{ is ordinal definable over } J_\beta(\mathbb{R})\}.$$

If $[\alpha, \beta]$ is weak, put

$$e(\Gamma) = \{A \subseteq \mathbb{R} \mid \exists \xi < \beta \text{ (A is ordinal definable over } J_\xi(\mathbb{R})\}.$$

So

$$\text{ENV}(\Gamma) = \bigcup_{z \in \mathbb{R}} e(\Gamma)(z)$$

is the boldface pointclass associated to $e(\Gamma)$.

A 0-suitable premouse is a minimal premouse N with one Γ Woodin, called δ^N. Such an N is A-iterable if it has a partial iteration strategy moving the $Col(\omega, \delta^N)$ term relation for A correctly. The reader should see [19] or [21] for full definitions. We have the following basic result of Woodin.

Theorem 38. *(Woodin) For any countable transitive set X, and A such that $A \in e(\Gamma)(z)$ for some $z \in X$, there is a 0-suitable, A-iterable premouse over X.*

See [14] for a full proof. [21] outlines a proof in the weak gap case. If $[\alpha, \beta]$ is weak, we let z_0 be a real parameter such that for some finite set F of ordinals, $\langle z_0, F\rangle$ satisfies a non-reflecting Σ_n type, where n is least such that $\rho_n(J_\beta(\mathbb{R})) = \mathbb{R}$. We let F_0 be the Brouwer-Kleene least such F, and let

$$J_\beta(\mathbb{R}) = \bigcup_n H_n$$

be the decompostion given in [?]. Thus in particular each H_n collapses to a member of $J_\beta(\mathbb{R})$. If $[\alpha, \beta]$ is not weak, let z_0 be a real such that for some Σ_1 formula $\varphi(v)$, we have

$$J_{\beta+1}(\mathbb{R}) \models \varphi[z_0] \text{ but } J_\beta(\mathbb{R}) \not\models \varphi[z_0].$$

Let ρ be a standard $Col(\omega, \omega_1^V)$ term for a real such that $\rho_g = z_0$, and let p_0 force all the properties of ρ we have enumerated so far. For $p \in Col(\omega, \omega_1^V)$ such that $p_0 \subseteq p$, let $g_p(n) = p(n)$ if $n \in dom(p)$, and $g_p(n) = g(n)$ otherwise. Let τ be a term for a real such that τ_g codes ρ_g and g in some natural way. It is easy to do this so that

(a) $z_0 \leq_T \tau_g$, and
(b) for all p, $\tau_g \equiv_T \tau_{g_p}$.

Put $z = \tau_g$. For any $A \in e(\Gamma)(z)$, put

$$\begin{aligned} B(A) \quad = \quad & \{(y,t) \in \mathbb{R} \times \mathbb{R} \mid y \text{ codes a countable, transitive } X \text{ such that} \\ & z \in X, \text{ and } t \text{ codes } \mathrm{Th}_\omega^N(X \cup \{X, \tau_A^N\}), \\ & \text{for some (all) 0-suitable, } A\text{-iterable } X\text{-premouse } N\ \}. \end{aligned}$$

Here τ_A^N is the standard $Col(\omega, \delta^N)$- term capturing A . "Some" is equivalent to "all" in the definition above because A-iterability yields an approximation to the comparison process which suffices to determine the theory in question. Note that $B(A) \in e(\Gamma)(z)$, because $e(\Gamma)(z)$ is closed under real quantification. By the scale analysis of [?], we have a self-justifying system $\mathcal{A} = \{A_i \mid i < \omega\}$ such that

(1) each A_i is in $e(\Gamma)(z)$,
(2) if $\alpha = \beta$ or $[\alpha, \beta]$ is strong, then for each $n < \omega$, $\mathrm{Th}_n^{J_\beta(\mathbb{R})}(\mathbb{R}) \in \mathcal{A}$
(3) if $[\alpha, \beta]$ is weak, then for all n, $\mathrm{Th}_\omega^{H_n}(\mathbb{R} \cup \{z, F_0\}) \in \mathcal{A}$, and
(4) for any n, $B(\langle A_i \mid i \leq n\rangle) \in \mathcal{A}$, where we regard $\langle A_i \mid i \leq n\rangle$ as a set of reals via some natural coding.

It is easy to also arrange that there is a fixed term $\dot{\mathcal{A}}$ such that

(5) for all $p \supseteq p_0$, $\dot{\mathcal{A}}^{g_p} = \dot{\mathcal{A}}^g$.

Let X be countable transitive, with $z \in X$. Let N^n be a 0-suitable, $\langle A_i \mid i \leq n\rangle$-iterable mouse over X. As in [19] we can simultaneously compare all the N^n to get a 0-suitable N over X such that N is $\langle A_i \mid i \leq n\rangle$-iterable for all n. But then condensation for term relations implies that N has a unique fullness preserving (ω_1, ω_1) iteration strategy which moves all the term relations $\tau^N_{A_i}$ correctly.[2] Put

$$Q(X) = \mathrm{Hull}^N(X \cup \{X\} \cup \{\tau^N_A \mid A \in \mathcal{A}\}.)$$

Condensation for term relations of a self-justifying system implies that $Q(X)$ has all the properties of N: it is 0-suitable, and has a unique fullness preserving (ω_1, ω_1) iteration strategy which moves all the term relations $\tau^{Q(X)}_{A_i}$ correctly. Moreover, $Q(X)$ is "sound", in that $Q(X) = \mathrm{Hull}^{Q(X)}(X \cup \{X\} \cup \{\tau^{Q(X)}_A \mid A \in \mathcal{A}\})$. Let

$$N_0 = Q(V_\omega \cup \{z\}),$$

and

$$N_{k+1} = Q(N_k),$$

and

$$N = \bigcup_{k<\omega} N_k.$$

Put $\delta^N_k = \delta^{N_k}$. We regard N as a premouse over z in the natural way. Note that because N_k is suitable, and hence Γ-full, no level of N_{k+1} projects across $o(N_k)$, and thus the δ_k are all Woodin in N.

Lemma 39. *There is a unique $\mathcal{A}$-guided strategy for N in $V[g]$.*

Proof. As in [19], there is a unique $\mathcal{A}$-guided iteration strategy Σ_0 for N_0. Let

$$i \colon N_0 \to S_0$$

be an iteration map by Σ_0. We can let i act on all of N, giving rise to

$$i \colon N \to S.$$

Put also $S_m = i(N_m)$, for all m. We do not yet know that S is even wellfounded, but in fact

Claim 40. For all m, $S_{m+1} = Q(S_m)$.

Proof. We prove it for $m = 0$. Let

$$W_k = \mathrm{Th}^{N_1}_\omega(N|\delta \cup \{\tau^{N_1}_{A_0}, ..., \tau^{N_1}_{A_k}\}),$$

where $\delta = \delta^N_0$. Note $W_k \in N_0$ because N_0 is Γ-full. Let

$$B = B(\langle A_0, ..., A_k \rangle).$$

[2]See [19]. The strategy chooses the limit over n of branches b_n moving all $\tau^N_{A_i}$ for $i \leq n$ correctly.

Now N_0 satisfies the sentence "it is forced in $Col(\omega, \delta)$ that if y codes $N|\delta$ and t codes W_k, then $(y,t) \in \tau_B^{N_0}$." Thus the same sentence is true of $i(\delta)$, $i(W_k)$, and $i(\tau_B^{N_0})$ in S_0. But $i(\tau_B^{N_0}) = \tau_B^{S_0}$, and so

$$\begin{aligned}\mathrm{Th}_\omega^{S_1}(S|i(\delta) \cup \{\tau_{A_0}^{S_1}, ..., \tau_{A_k}^{S_1}\}) &= i(W_k) \\ &= \mathrm{Th}_\omega^{Q(S_0)}(S|i(\delta) \cup \{\tau_{A_0}^{Q(S_0)}, ..., \tau_{A_k}^{Q(S_0)}\}).\end{aligned}$$

It follows that there is a natural isomorphism between

$$\mathrm{Hull}_\omega^{S_1}(S|i(\delta) \cup \{\tau_{A_0}^{S_1}, ..., \tau_{A_k}^{S_1}\})$$

and

$$\mathrm{Hull}_\omega^{Q(S_0)}(S|i(\delta) \cup \{\tau_{A_0}^{Q(S_0)}, ..., \tau_{A_k}^{Q(S_0)}\}).$$

Moreover, these isomorphisms commute with the inclusion maps on the hulls, because they are determined by the $i(W_k)$. Finally, S_1 and $Q(S_0)$ are the unions of the respective sequences of hulls, as k varies. (In the case of S_1, this is because $N_1 = Q(N_0)$, and i came from an iteration based on N_0.) Thus $S_1 \cong Q(S_0)$. The proof for $m > 0$ is the same. □

But now $S_1 = Q(S_0)$ has a unique iteration strategy Σ_1 for trees above S_0. Letting $i \colon S \to T$ come from an iteration of S_1 by this strategy, and $T_m = i(S_m)$ we get $T_{m+1} = Q(T_m)$ for all $m \geq 1$ by repeating the proof of the claim above. We can then move on to iterating T_2 above T_1, etc. Clearly, this describes an iteration strategy for N acting on normal trees.[3] □

N is a mouse over τ_g, but it can be re-arranged as a mouse over τ_{g_p} whenever $p \supseteq p_0$. The re-arranged mouse has the same universe and extender sequence; it just has a different (but Turing equivalent) real distinguished at the bottom. What is more, we have a fixed term $\dot{N}$ such that for all $p \supseteq p_0$, $\dot{N}_{g_p}$ is the re-arrangement of N as a mouse over τ_{g_p}. This is because of the symmetry in the construction of N, and in particular, because $\dot{\mathcal{A}}_g = \dot{\mathcal{A}}_{g_p}$ for all such p. This enables us to build in V a premouse N_τ over τ such that $N_\tau[g] = N$. We construct $N_\tau|\alpha$ by induction on α, maintaining that

$$(N_\tau|\alpha)[g] = N|\alpha,$$

along with the correspondence of projecta and parameters. α is active in N_τ iff it is active in N, and if so,

$$E_\alpha^{N_\tau} = E_\alpha^N \cap N_\tau|\alpha.$$

[3] In fact our strategy applies to trees of the form: a stack of normal trees below the first Woodin, then a stack of normal trees between the first and second Woodin, then a stack between the second and third, etc.

$N_\tau|(\alpha+1) \in V$ because by induction, $N_\tau|\alpha \in V$, and because E^N_α is independent of g. $E^{N_\tau}_\alpha$ is an extender over N_τ because g was generic over V, and the forcing is small. The reader can find all the details of this construction in [21] and [**?**]. Let Σ^g be the unique iteration strategy for N given by the Lemma. Iterating N_τ is the same as iterating $N_\tau[g] = N$, because the forcing is small, and thus we can regard Σ^g as a strategy for N_τ. Moreover Σ, which denotes $\Sigma^g \upharpoonright V$, is in V by the symmetry in our construction of Σ^g. Since Σ^g condenses well, Σ condenses well. We have finished steps 1 and 2 of the general plan.

We now execute step 3. Here we use $\mathsf{WRP}_{(2)}(\omega_2)$ in V to extend our ω_2-iteration strategy to an ω_3-iteration strategy. In fact, simultaneous stationary reflection for pairs of subsets of ω_2 is enough.

Lemma 41. *Let M be a premouse of cardinality $\leq \omega_1$, and let Σ be any ω_2 iteration strategy for M which condenses well. Suppose that whenever $S, T \subseteq \omega_2$ are stationary and consist of ordinals of countable cofinality, there is a $\nu < \omega_2$ such that S and T are stationary in ν. Then there is a unique ω_3 iteration strategy Ω for M such that*

(1) $\Sigma \subseteq \Omega$, *and*
(2) Ω *condenses well.*

Proof. We omit the easy proof that there is at most one such Ω. Fix η large. Let $\mathcal{T}$ be an iteration tree on M with $\mathrm{lh}(\mathcal{T}) < \omega_2$. We say $\langle X_\alpha \mid \alpha < \omega_2 \rangle$ is a $\mathcal{T}$-*chain* iff

(a) $X_\alpha \prec V_\eta$, for all $\alpha < \omega_2$,
(b) $M, \mathcal{T} \in X_0$, and
(c) $|X_\alpha| = \omega_1$, and $X_\alpha \cap \omega_2 \in \omega_2$, for all $\alpha < \omega_2$.

Given a $\mathcal{T}$-chain $\vec{X}$, we let $\pi_\alpha \colon H_\alpha \cong X_\alpha$ with H_α transitive, let $\pi_{\alpha,\beta} = \pi_\beta^{-1} \circ \pi_\alpha$, and let $\mathcal{T}_\alpha = \pi_\alpha^{-1}(\mathcal{T})$. We say that $\vec{X}$ is Σ-*good* iff each $\mathcal{T}_\alpha$ is by Σ, and in that case, we set

$$b_\alpha = \Sigma(\mathcal{T}_\alpha)$$

for all $\alpha < \omega_2$. There is of course no reason that we should have $b_\alpha \in H_\alpha$.

Claim 42. Let $\vec{X}$ be a Σ-good $\mathcal{T}$-chain, and let $\gamma < \omega_2$ with $cof(\gamma) = \omega_1$; then for club many $\alpha < \gamma$, $\pi_{\alpha,\gamma}``b_\alpha \subseteq b_\gamma$.

Proof. We take cases on the cofinality of the length of $\mathcal{T}$. Suppose first $cof(lh(\mathcal{T})) = \omega$. Then for all sufficiently large $\alpha < \gamma$, $ran(\pi_{\alpha,\gamma})$ is cofinal in b_γ, and thus applying condensation to the support-closed subtree of $\mathcal{T}_\gamma{}^\frown b_\gamma$ determined by $ran(\pi_{\alpha,\gamma})$, we get that $\pi^{-1}_{\alpha,\gamma}``b_\gamma = \Sigma(\mathcal{T}_\alpha) = b_\alpha$. So the desired club is just a tail below γ. Suppose $cof(lh(\mathcal{T})) = \omega_1$.

Then $cof(lh(\mathcal{T}_\xi)) = \omega_1$, for all ξ. Also, for all $\alpha < \gamma$, $\pi_{\alpha,\gamma}$"b_α is cofinal in $lh(\mathcal{T}_\gamma)$. Since $\mathcal{T}_\gamma$ has at most one cofinal branch, we get $\pi_{\alpha,\gamma}$"$b_\alpha \subseteq b_\gamma$. Finally, suppose $cof(lh(\mathcal{T})) = \omega_2$. As in case two, $cof(lh(\mathcal{T}_\xi)) = \omega_1$, for all ξ, but now, $\alpha < \gamma \Rightarrow \pi_{\alpha,\gamma}$ is discontinuous at $lh(\mathcal{T}_\alpha)$. Fixing γ with $cof(\gamma) = \omega_1$, we can find club many $\alpha < \gamma$ such that $ran(\pi_{\alpha,\gamma}) \cap b_\gamma$ is cofinal in $\sup \pi_{\alpha,\gamma}$"$lh(\mathcal{T}_\alpha)$. For any such α, condensation for the support-closed subtree of $\mathcal{T}_\gamma{}^\frown b_\gamma$ determined by $ran(\pi_{\alpha,\gamma})$, implies that $\pi_{\alpha,\gamma}^{-1}$"$b_\gamma = \Sigma(\mathcal{T}_\alpha) = b_\alpha$. □

Let $\vec{X}$ be a Σ-good $\mathcal{T}$-chain. We say $\vec{X}$ is *coherent* if and only if whenever $\alpha < \gamma < \omega_2$, then $\pi_{\alpha,\gamma}$"$b_\alpha \subseteq b_\gamma$. In this case, we say $\vec{X}$ *justifies* b, where

$$b = \bigcup_{\alpha<\omega_2} b_\alpha.$$

It is easy to see

Claim 43. $\mathcal{T}$ has at most one branch b which is justified by some coherent $\mathcal{T}$-chain.

Proof. If $\vec{X}$ and $\vec{Y}$ are Σ-good $\mathcal{T}$-chains, then for club many $\alpha < \omega_2$, $X_\alpha \cap \omega_2 = Y_\alpha \cap \omega_2$. Thus for club many $\alpha < \omega_2$, $\mathcal{T}_\alpha^{\vec{X}} = \mathcal{T}_\alpha^{\vec{Y}}$ and $b_\alpha^{\vec{X}} = b_\alpha^{\vec{Y}}$. □

So we define

$$\Omega(\mathcal{T}) = b \Leftrightarrow \text{ b is justified by some coherent } \mathcal{T}\text{-chain.}$$

Claim 44. If $\mathcal{T}$ is by Ω, then every $\mathcal{T}$-chain is Σ-good.

Proof. Let $\mathcal{T}$ be of minimal length such that the claim is false. Suppose first that $lh(\mathcal{T})$ is a limit ordinal. Let $\vec{X}$ be a $\mathcal{T}$-chain. If $\alpha < \gamma < \omega_2$, then $\mathcal{T}_\alpha$ is the collapse of a support-closed subtree of $\mathcal{T}_\gamma$, so since Σ condenses well, we have that $\mathcal{T}_\gamma$ is not by Σ for all sufficiently large $\gamma < \omega_2$. Using a surjective map $f\colon \omega_2 \to \mathrm{lh}(\mathcal{T})$ with $f \in X_0$, and a Fodor argument, we can fix $\xi < lh(\mathcal{T})$ such that for stationary many $\alpha < \omega_2$, $\pi_\alpha^{-1}(\mathcal{T} \restriction \xi)$ is not by Σ. But $\vec{X}$ is a $\mathcal{T} \restriction \xi$-chain, contrary to the minimality of $lh(\mathcal{T})$. Thus $lh(\mathcal{T}) = \lambda + 1$ for some λ. It is clear that λ must be a limit ordinal. Let $b = \Omega(\mathcal{T} \restriction \lambda$, and let $\vec{X}$ be a $\mathcal{T}$-chain. Let $\vec{Y}$ be a $\mathcal{T} \restriction \lambda$-chain which justifies b. There are club many $\alpha < \omega_2$ such that $X_\alpha \cap lh(\mathcal{T}) = Y_\alpha \cap lh(\mathcal{T})$, and for such α, $(\pi_\alpha^{\vec{X}})^{-1}(b) = (\pi_\alpha^{\vec{Y}})^{-1}$"$b = b_\alpha^{\vec{Y}}$. Thus for club many α, $\mathcal{T}_\alpha^{\vec{X}}$ is by Σ. Condensation implies this is true for all α. This contradiction completes the proof. □

Claim 45. $\Sigma \subseteq \Omega$, and Ω condenses well.

Proof. If $\mathcal{T}$ is by Σ, then in any $\mathcal{T}$-chain, we have $\mathcal{T}_\alpha = \mathcal{T}$ for all $\alpha < \omega_2$, so every $\mathcal{T}$-chain justifies $\Sigma(\mathcal{T})$. For condensation, suppose $\Omega(\mathcal{T}) = b$, and $\mathcal{U}^\frown c$ is the collapse of a support-closed subtree of $\mathcal{T}^\frown b$, and $\Omega(\mathcal{U}) = d$ where $d \neq c$. It is easy to see that there is a single $\vec{X}$, with $\mathcal{T}, \mathcal{U}, b, c, d \in X_0$, which justifies both b and d. But this gives a failure of condensation for Σ. □

Claim 46. Suppose $\mathcal{T}$ is by Ω, and $lh(\mathcal{T}) < \omega_3$; then there is a b such that $\Omega(\mathcal{T}) = b$.

Proof. Fix any $\xi < lh(\mathcal{T})$, and any $\mathcal{T}$-chain $\vec{X}$. Since $\vec{X}$ is Σ-good, we have $b_\alpha = \Sigma(\mathcal{T}_\alpha)$ for $\alpha < \omega_2$. We claim that exactly one of the following holds:

(1) for ω-club many $\alpha < \omega_2$, $\pi_\alpha^{-1}(\xi) \in b_\alpha$,
(2) for ω-club many $\alpha < \omega_2$, $\pi_\alpha^{-1}(\xi) \notin b_\alpha$.

It is clear that not both can hold. Suppose both fail. Let S be the stationary set of α of cofinality ω where $\xi \in ran(\pi_\alpha)$ and (1) fails, and T the stationary set of α of cofinality ω where (2) fails. By our stationary reflection hypothesis, we can fix γ of cofinality ω_1 such that both S and T are stationary in γ. Note $\xi \in ran(\pi_\gamma)$. If $\pi_\gamma^{-1}(\xi) \in b_\gamma$, then by the first claim, $\pi_\alpha^{-1}(\xi) \in b_\alpha$ for club-in-γ many α, so T was not stationary in α, contradiction. Similarly, if $\pi_\gamma^{-1}(\xi) \notin b_\gamma$, then The first claim implies S is not stationary in γ, a contradiction. So at least one of (1) and (2) holds. It also implies that the ω-clubs of (1) and (2) can be taken to be fully club in ω_2. Define b by:

$$\xi \in b \Leftrightarrow \text{ for club many } \alpha < \omega_2,\ \pi_\alpha^{-1}(\xi) \in b_\alpha.$$

Taking a diagonal intersection, we can find a club $C \subseteq \omega_2$ such that for $\alpha \in C$, π_α“$b_\alpha \subseteq b$. But then $\langle X_\alpha \mid \alpha \in C\rangle$ is a coherent $\mathcal{T}$-chain which justifies b. □

This completes the proof of Lemma 41. □

Applying Lemma 41, let us use Σ to denote the unique ω_3 iteration strategy for N_τ which condenses well, and extends $\Sigma^g \restriction V$. Proceeding to step 4, we need to further extend Σ so that it acts on all trees in $H(\omega_1)^{V[g][h]}$, whenever h is $V[g]$-generic over some poset in $H(\omega_2)^{V[g]}$. These extensions of Σ will be mutually consistent. At the same time, we will be showing that the gap $[\alpha, \beta]$ of $V[g]$ has counterparts in every $V[g][h]$. The following little lemma will be useful.

Lemma 47. *Let Γ be an iteration strategy for S which condenses well. Let $\pi\colon R \to S$ be sufficiently elementary that the pullback strategy Γ^π for R exists; then Γ^π also condenses well.*

Proof. Let $\mathcal{T}$ be a tree according to Γ^π, and $\mathcal{U}$ a support-closed subtree of $\mathcal{T}$ corresponding to those $\mathcal{M}_\alpha^{\mathcal{T}}$ such that $\alpha \in X$, and let $\bar{\mathcal{U}}$ be the collapse of $\mathcal{U}$. It is easy to see the lifted tree $\pi\bar{\mathcal{U}}$ is the collapse of the support-closed subtree of $\pi\mathcal{T}$ corresponding to those $\mathcal{M}_\alpha^{\pi\mathcal{T}}$ such that $\alpha \in X$. Since Γ condenses well, $\pi\bar{\mathcal{U}}$ is according to Γ, and hence $\bar{\mathcal{U}}$ is according to Γ^π. □

We need to use hybrid strategy mice. Suppose Ω is an iteration strategy for some structure M, and Ω condenses well. Let A be transitive, with $M \in A$. We obtain a hybrid Ω-premouse by adding extenders with critical points above A to a coherent sequence we are building, and at the same time closing the model we are building under Ω, and giving it a predicate for Ω. The construction can only go on as long as all (non-dropping) iteration trees according to Ω we construct are in the domain of Ω. (M may or may not be a fine-structural premouse, but in any case, it is convenient to only close under Ω on non-dropping trees.) We refer the reader to [19] for a brief discussion of such Ω-hybrids, and to [14] for a more thorough one.

Definition 48. Let Ω be an $|A|^+$-iteration strategy for M which condenses well, where A is transitive and $M \in A$; then $P_n^\Omega(A)^\sharp$ is the minimal $|A|^+$-iterable hybrid Ω- mouse over A which is active, and satisfies "there are n Woodin cardinals".

We note that the iterations referred to here all leave A, and hence M, fixed. It is part of iterability that they must move Ω to itself. One can hope to construct such iterable hybrid mice in a K^c construction, because Ω condenses well, and hence Ω will condense to itself under realizing maps. The iterability demand we have made for $P_n^\Omega(A)^\sharp$ in 48 is the minimal one which guarantees uniqueness, granted that $H(|A|^+)$ is closed under P_{n-1}^Ω-sharps. We shall never consider a putative $P_n^\Omega(A)$-sharp unless we already know $H(|A|^+)$ is closed under P_{n-1}^Ω-sharps. In practice, we often have more iterability than the minimal demand. Our core-model-induction proof that $H(\omega_3)$ of V is closed under the $M_n^\sharp$ operators generalizes routinely to hybrid mouse operators, and gives:

Lemma 49. *Assume NS is saturated, and* $\mathsf{WRP}_{(2)}(\omega_2)$ *holds. Let* $S \in H(\omega_1)$*, and let* Ω *be an* ω_3*-iteration strategy for* S *which condenses well. Then for all transitive* $A \in H(\omega_3)$ *such that* $S \in A$*, and all* $n < \omega$*,* $P_n^\Omega(A)^\sharp$ *exists and is* ω_3*-iterable.*

We will eventually apply this lemma; it is the place where core model theory gives us new mice. However, we need more preliminary work before we are ready to use it.

Lemma 50. *For all $A \in H(\omega_3)$, $P_0^\Sigma(A)^\sharp$ exists and is ω_3-iterable.*

Proof. We show first that $P_0^\Sigma(A)^\sharp$ exists for all such $A \in H(\omega_2)$, then extend this to $A \in H(\omega_3)$ using simultaneous reflection. Let $G \subset P(\omega_1)/NS$ be V-generic and let

$$i\colon V \to M \subseteq V[G]$$

be the generic embedding. Since $N_\tau \in H(\omega_2)^V$, we have $N_\tau \in M$, and $i \restriction N_\tau \in M$. So inside M, we can form the $(i \restriction N_\tau)$-pullback of $i(\Sigma)$, which we denote by $i(\Sigma)^i$. From the point of view of M, $i(\Sigma)^i$ is an ω_3 iteration strategy for N_τ, and by 47, it condenses well in M.

Claim 51. $i(\Sigma)^i$ agrees with Σ on all trees in the intersection of the two domains.

Proof. We first consider trees in $H(\omega_2)^V$, all of which are in both domains. Let $\mathcal{T} \in H(\omega_2)^V$ be a tree according to Σ. Note that $i \restriction N_\tau \in M$. In M the copied tree $i\mathcal{T}$ on $i(N_\tau)$ is satisfied to be the collapse of a support closed subtree $i(\mathcal{T})$. Since $i(\Sigma)$ condenses well in M, $i\mathcal{T}$ is according to $i(\Sigma)$. Hence $\mathcal{T}$ is according to the pull-back $i(\Sigma)^i$. Now let $\mathcal{U}$ be a tree in V of size ω_2^V in V which is according to both Σ and $i(\Sigma)^i$, and is of limit length. Let

$$b = i(\Sigma)^i(\mathcal{U})$$

and

$$c = \Sigma(U)$$

and

$$b \neq c.$$

Note that $cf(lh(\mathcal{U}))$ must be countable in $V[G]$. In V, we can write $\mathcal{U} = \bigcup_{\alpha<\omega_2} \mathcal{U}_\alpha$, where this is an increasing continuous chain of support-closed subtrees, each of size ω_1.. Going to V[G], where $cf(lh(\mathcal{U}))$ is countable, we see that

$$b \cap \mathcal{U}_\alpha \text{ is cofinal in } \mathcal{U}_\alpha$$

and

$$c \cap \mathcal{U}_\alpha \text{ is cofinal in } \mathcal{U}_\alpha,$$

for all sufficiently large α, so we may assume all α. Let $\bar{\mathcal{U}_\alpha}$ denote the collapse of $\mathcal{U}_\alpha$, $\bar{b}_\alpha$ the collapse of $b \cap \mathcal{U}_\alpha$, $\bar{c}_\alpha$ the collapse of $c \cap \mathcal{U}_\alpha$. Fix α such that $\bar{b}_\alpha \neq \bar{c}_\alpha$. Now

$$\bar{c}_\alpha = \Sigma(\bar{\mathcal{U}_\alpha})$$

because Σ condenses well in V. On the other hand,

$$\bar{b}_\alpha = i(\Sigma)^i(\bar{\mathcal{U}_\alpha}),$$

because $i(\Sigma)^i$ condenses well in M. Since Σ and $i(\Sigma)^i$ must agree at $\bar{U}_\alpha$ by the first part, we are done. $\square$

Now fix a transitive $A \in H(\omega_2)$ such that $N_\tau \in A$. Let $L^\Sigma[A]$ be the minimal model of height ω_3 which has A as a member and is closed under Σ, and is expanded by a predicate for Σ. In M, we can form $L^{i(\Sigma)^i}[A]$ in parallel fashion. By 51, these two models are the same. So $L^\Sigma[A] \in M$. But $i(NS)$ is saturated in M, so by the same argument that shows that the existence of a saturated ideal implies $0^\sharp$ exists, we get some $P \in M$ such that

$$M \models P = P_0^\Sigma(A)^\sharp.$$

Being $P_0^\Sigma(A)^\sharp$ is a first order property, combined with linear iterability by the last (and only) extender in a way that moves Σ to itself, for iterations of length $< \omega_3^M = \omega_3^V$. But now let h be V-generic for $Col(\omega, \omega_2^V)$, and such that $G \in V[h]$. The required iterability of P is upward absolute, that is

$$V[h] \models P \text{ is } \omega_3^V \text{ iterable},$$

so since ω_3^V is still uncountable in $V[h]$,

$$V[h] \models P = P_0^\Sigma(A)^\sharp.$$

By the homogeneity of $Col(\omega, \omega_2)$, $P \in V$, and

$$V \models P = P_0^\Sigma(A)^\sharp,$$

and we are done in the case $A \in H(\omega_2)$. We now use $\mathsf{WRP}_{(2)}(\omega_2)$, via hybrid mouse reflection at ω_2, to show that $P_0^\Sigma(A)^\sharp$ exists, for all transitive $A \in H(\omega_3)$ such that $N_\tau \in A$. Without loss of generality, let us assume $A \subseteq \omega_2$. Let ϕ be a formula in the language of set theory together with a predicate symbol $\dot{\Sigma}$ and constant symbol $\dot{A}$, and let $\vec{\alpha} \in \omega_2^{<\omega}$. For $\sigma \prec H(\omega_2)$ countable, let

$$\pi_\sigma \colon M_\sigma \to H_{\omega_2}$$

be the transitive collapse, and $A_\sigma = \pi_\sigma^{-1}(A_\sigma)$, $N_\sigma = \pi_\sigma^{-1}(N_\tau)$, and $\vec{\alpha}_\sigma = \pi_\sigma^{-1}(\vec{\alpha})$. Note that for and such σ, the pullback strategy Σ^{π_σ} is a full ω_3 iteration strategy for N_σ, and it condenses well. Using our saturated ideal, we then have that

$$P_0^{\Sigma^{\pi_\sigma}}(A_\sigma)^\sharp \text{ exists},$$

and is ω_3 iterable. We put

$$(\phi, \vec{\alpha}) \in P_0^\Sigma(A)^\sharp \quad \Leftrightarrow \quad \text{for club many } \sigma \in P_{\omega_1}(H_{\omega_2}) \\ (\phi, \vec{\alpha}_\sigma) \in P_0^{\Sigma^{\pi_\sigma}}(A_\sigma)^\sharp.$$

(Here we identify the structure $P_0^{\Sigma}(A)^\sharp$ with its theory with parameters from ω_2.) In order to see that this definition works, we must show that every $(\phi, \vec{\alpha})$ is decided on a club. So suppose neither $(\phi, \vec{\alpha})$ nor $(\neg\phi, \vec{\alpha})$ is in $P_0^{\Sigma}(A)^\sharp$ according to the definition above. As usual, we find a transitive $X \prec H_{\omega_2}$ with $|X| = \omega_1$ such that both sets are stationary in $P_{\omega_1}(X)$. Without loss of generality, assume $\vec{\alpha}, N_\tau \in X$, and

$$(\phi, \vec{\alpha}) \in P_0^{\Sigma}(A \cap X)^\sharp.$$

It is then easy to see that for club many $\sigma \in P_{\omega_1}(X)$, $(\phi, \vec{\alpha}_\sigma) \in P_0^{\Sigma^{\pi_\sigma}}(A_\sigma)^\sharp$. That is because for club many $\sigma \prec X$, $\sigma = Z \cap X$ for some $Z \prec V_\eta$ with $P_0^{\Sigma}(A \cap X)^\sharp \in Z$. Letting $\pi \supseteq \pi_\sigma$ be the collapse of Z, we get that

$$\pi^{-1}(P_0^{\Sigma}(A \cap X)^\sharp) = P_0^{\Sigma^{\pi_\sigma}}(A_\sigma)^\sharp.$$

To see this, note $\pi^{-1}(\Sigma) \subseteq \Sigma^{\pi_\sigma}$ by our argument in the first part of the proof of 50. So the strategy predicate in $\pi^{-1}(P_0^{\Sigma}(A \cap X)^\sharp)$ denotes Σ^{π_σ}. Moreover, iterates S of $\pi^{-1}(P_0^{\Sigma}(A \cap X)^\sharp)$ embed into iterates S^* of $P_0^{\Sigma}(A \cap X)^\sharp$, and the strategy predicate of S^* denotes a fragment of Σ, so the strategy predicate of S denotes a fragment of Σ^{π_σ}. So we have shown $(\phi, \vec{\alpha}) \in P_0^{\Sigma}(A)^\sharp$, or $(\neg\phi, \vec{\alpha}) \in P_0^{\Sigma}(A)^\sharp$. This easily gives that our $P_0^{\Sigma}(A)^\sharp$ has the first order properties required of $P_0^{\Sigma}(A)^\sharp$.

We must see that its strategy predicate denotes Σ, and that linear iterates of it move Σ correctly. Let I be a linear iteration of length $< \omega_3$ of P, with last model Q such that $\dot{\Sigma}^Q \not\subseteq \Sigma$. We can find

$$\pi \colon H \to V_\eta$$

such that M is countable transitive, and everything relevant is in $\operatorname{ran}(\pi)$. Because $\operatorname{ran}(\pi) \cap \omega_2$ meets the clubs definable over V_η from elements of $\operatorname{ran}(\pi)$, we get

$$\pi^{-1}(P) = P_0^{\Sigma^\pi}(\pi^{-1}(A))^\sharp.$$

Also, $\pi^{-1}(\Sigma) = \Sigma^\pi \cap H$. So $\dot{\Sigma}^{\pi^{-1}(N)} \not\subseteq \Sigma^\pi$. This contradicts the fact that linear iterations of $P_0^{\Sigma^\pi}(\pi^{-1}(A))^\sharp$ *do* move Σ^π to itself, by definition. □

It is tempting to use Lemma 49 to conclude that $H_(\omega_3)$ is closed under the P_n^{Σ}-sharp operator. This would work if $N_\tau \in H(\omega_1)^V$, and would essentially finish our proof. However, in general we only have N_τ countable in NS-generic ultrapower, and what we get is

Lemma 52. *Let $G \subset P(\omega_1)/NS$ be V-generic, and $i \colon V \to M$ the generic embedding. Then the following is true in M: for all transitive*

$A \in H(\omega_3)$ *such that* $N_\tau \in A$, *and all* $n < \omega$, $P_n^{i(\Sigma)^i}(A)^\sharp$ *exists, and is* ω_3*-iterable.*

We need to relate the hybrids in M given by this lemma to hybrids in $V[g]$. This involves showing $i(\Sigma)^i$ is consistent with a natural extension of Σ acting on $V[g][G]$. Creating this extension is basically the same as lifting the gap $[\alpha, \beta]$ of $V[g]$ to $V[g][G]$.

We shall use genericity iterations of N_τ to lift $J_\beta(\mathbb{R})^{V[g]}$ and Σ^g to $V[g][h]$, for any h generic over $V[g]$ for a poset in $H(\omega_2)^{V[g]}$. To this end, recall our self-justifying system $\mathcal{A} = \{A_i \mid i < \omega\}$ in $V[g]$. For $A \in \mathcal{A}$ and δ a Woodin cardinal of N, we have the $Col(\omega, \delta)$-term $\tau^N_{A,\delta}$, whose images in Σ^g-iterations always capture A. Since $N = N_\tau[g]$, we have $\tau^N_{A,\delta} = \rho_g$ for some $Col(\omega, \omega_1^V)$- term ρ. Let $\sigma_{A,\delta}$ be the canonical $Col(\omega, \omega_1^V) \times Col(\omega, \delta)$ term such that for all generics $k \times l$,

$$(\sigma_{A,\delta})_{k \times l} = (\rho_k)_l.$$

Thus

$$(\sigma_{A,\delta})_{g \times l} = (\tau^N_{A,\delta})_l,$$

for l being $Col(\omega, \delta)$ generic over $N_\tau[g]$.

Lemma 53. *Let* $h \subset \mathbb{P}$ *be* $V[g]$*-generic where* $\mathbb{P} \in (H(\omega_3^V))^{V[g]}$. *Then in* $V[g][h]$ *there are*

(1) *sets* $A_i^* \subseteq \mathbb{R}$ *such that*

$$(HC^{V[g]}, \in, A_i)_{i<\omega} \prec (HC^{V[g][h]}, \in, A_i^*)_{i<\omega},$$

(2) *an ordinal* β^h *and embedding*

$$\pi \colon J_\beta(\mathbb{R})^{V[g]} \to J_{\beta^h}(\mathbb{R})^{V[g][h]}$$

such that π *is fully elementary if* $\alpha = \beta$ *or* $[\alpha, \beta]$ *is strong, and* π *is* Σ_n*-elementary for* n *least such that* $\rho_n(J_\beta(\mathbb{R})^{V[g]}) = \mathbb{R}$ *otherwise, and*

(3) *a unique* ω_3^V*-iteration strategy* $\Sigma^h \in V[g][h]$ *for* N *which extends* $\Sigma^g \cup \Sigma$ *and condenses well.*

Proof. We begin with (1). A_i^* comes from interpreting the images of $\tau^{N_\tau}_{A_i}$ under genericity iterations. Note first

Claim 54. In V, let $M \in H_{\omega_3}$ be any non-dropping Σ-iterate of N_τ, and let $k < \omega$. Let $x \in \mathbb{R} \cap V[g][h]$; then there is (in V) a Σ-iteration map $i \colon M \to P$ with $crit(i) > \delta_k^M$ such that for any $Col(\omega, \delta_k^M)$-generic l over P such that $l \in V[g][h]$ and $g, h \in P[l]$, we have that $x \in P[l][f]$, for some $f \in V[g][h]$ such that f is $Col(\omega, \delta_{k+1}^P)$ -generic over P.

Proof. Let $x = \sigma_{g\star h}$. Working in V, we use the standard genericity iteration for the ω_2^V-generator version of the extended algebra of M at δ_{k+1}^M to make σ generic. We get in V an $i\colon M \to P$ with $crit(i) > \delta_k^M$ such that for any $Col(\omega, \delta_k^M)$-generic l over P, there is f as in our claim with $\sigma \in P[l][f]$. So if $g, h \in P[l]$, then $x \in P[l][f]$. The important thing to note is that the genericity iteration yielding i terminates. This follows from the fact that $P^\Sigma(C)^\sharp$ exists, where $C \in H_{\omega_3}$ codes up σ and the iteration from N_τ to M. □

Claim 55. Let $i\colon N_\tau \to P$ and $j\colon N_\tau \to Q$ be non-dropping Σ-iterations of N_τ (in V), and let δ and μ be Woodin cardinals of N_τ. Let $A \in \mathcal{A}$ be in our self-justifying system from $V[g]$. Let $x \in \mathbb{R}^{V[g][h]}$ be such that

$$x \in P[g][l_0] \cap Q[g][l_1],$$

where l_0, l_1 are generic over $P[g], Q[g]$ for the collapses of $i(\delta)$ and $j(\mu)$, respectively. Then

$$x \in i(\sigma_{A,\delta})_{g\times l_0} \Leftrightarrow x \in j(\sigma_{A,\mu})_{g\times l_1}.$$

Proof. If not, we have $(p, q) \in g \star h$ such that

$$(p, q) \Vdash \phi(\check{N}_\tau, \check{\Sigma}),$$

where ϕ in the language for forcing over V expresses the failure of out claim in $V[g][h]$. Here ϕ also involves check-names for $\sigma_{A,\delta}$ and $\sigma_{A,\mu}$, which we have suppressed. In V, let

$$\pi\colon H \to V_\eta,$$

where H is transitive and of size ω_1, $\omega_1 \in H$, and everything relevant is in $ran(\pi)$. We can find

$$\bar{h} \in V[g]$$

so that

$$g \star \bar{h} \text{ is } Col(\omega, \omega_1) \star \pi^{-1}(\dot{\mathbb{P}})\text{-generic over } H,$$

and $\pi^{-1}(q) \in \bar{h}$. Note that by condensation for Σ,

$$\pi^{-1}(\Sigma) \subseteq \Sigma.$$

But then, the fact that $H[g][\bar{h}] \models \phi[N_\tau, \pi^{-1}(\Sigma)]$ yields a Σ^g-iteration of N which fails to move one of the term relations for A correctly. This is a contradiction. □

Motivated by these claims, working in $V[g][h]$ we put for $x \in \mathbb{R}$ and $A \in \mathcal{A}$,

$$\begin{aligned} x \in A^h \quad \Leftrightarrow \quad & \exists i \exists \delta \exists l (i\colon N_\tau \to P \text{is a } \Sigma\text{-iteration and} \\ & l \text{ is } P[g]\text{-generic and } x \in i(\sigma_{A,\delta})_{g\times l}). \end{aligned}$$

It is easy to see that

$$A^h \cap V[g] = A,$$

because the iteration given by 54 can be taken in $H^V_{\omega_2}$ in this case, and such iterations correspond to iterations by Σ^g, which moves $\tau^N_{A,\delta}$ correctly. Note $\mathcal{A}$ is closed under real quantification. Fixing i, we have a j such that

$$V[g] \models \forall \vec{x} \in \mathbb{R}^{<\omega}(A_j(\vec{x}) \Leftrightarrow \exists y A_i(\vec{x}, y)).$$

But this fact is coded into the first order theory over N_τ of the term relations $\sigma_{A_i,\delta}$ and $\sigma_{A_j,\mu}$. More precisely, given $\delta < \mu$ Woodins of N_τ, there is a $p \in g$ which forces over N_τ the statement "whenever k is $N_\tau[\dot{g}]$-generic over $Col(\omega, \delta)$ and $\vec{x} \in N_\tau[\dot{g}][k]$, then $\vec{x} \in (\sigma_{A_j,\delta})_{\dot{g}\times k}$ if and only if 1 forces in $Col(\omega, \mu)$ over $N_\tau[\dot{g}][k]$ " there is a y such that $(\vec{x}, y) \in (\sigma_{A_i,\mu})_{\dot{g}\times t}$, where t is the re-arrangement of $k \times \dot{G}$."" These first order facts are preserved by our genericity iterations of N_τ, and those are sufficiently numerous by 54, and coherent in how they move the $\sigma_{A,\nu}$ by 55, that we get

$$V[g][h] \models \forall \vec{x} \in \mathbb{R}^{<\omega}(A^h_j(\vec{x}) \Leftrightarrow \exists y A^h_i(\vec{x}, y)).$$

We leave any further calculation here to the reader.[4] Also, for any i there is a j such that

$$V[g] \models \forall \vec{x}(A_i(\vec{x} \Leftrightarrow \neg A_j(\vec{x})).$$

Fixing such i, j, we then have

$$V[g][h] \models \forall \vec{x}(A^h_i(\vec{x} \Leftrightarrow \neg A^h_j(\vec{x})).$$

Generalizing slightly, we get that for any formula ϕ in the language of our two structures, there is a $j = j_\phi$ such that for all $\vec{x}$ in $V[g]$,

$$((\mathrm{HC}^{V[g]}, \in, A_i)_{i<\omega} \models \phi[\vec{x}]) \Leftrightarrow A_j(\vec{x}),$$

and for all $\vec{x}$ in $V[g][h]$

$$((\mathrm{HC}^{V[g][h]}, \in, A^h_i)_{i<\omega} \models \phi[\vec{x}]) \Leftrightarrow A^*_j(\vec{x}).$$

Since $A_j = A^h_j \cap V[g]$, we are done with part (1). Part (2) of the theorem follows easily from part (1), and the fact that the A_i code the appropriate fragments of the theory of $J_\beta(\mathbb{R})$. It is routine then use the A^h_i to construct a structure over $\mathbb{R} \cap V[g][h]$ into which $J_\beta(\mathbb{R})^{V[g]}$ embeds with the required degree of elementarity. One need only show the structure over $\mathbb{R} \cap V[g][h]$ one gets is well-founded. The proof of

[4]See [?] for a similar argument. It was to make this argument possible that we moved to an N with ω Woodiins, rather than just one.

this is a reflection argument very similar to the proof of Claim 2, so we omit it. We let β^h be such that $\omega\beta^h$ is the height of this structure.

Finally, we turn to (3). By part (2), β^h ends a gap $[\alpha^h, \beta^h]$ in $V[g][h]$, and the A_i^h constitute a self-justifying system which seals this gap. We claim that the A_i^h guide an iteration strategy Σ^h for N_τ, or equivalently for $N = N_\tau[g]$, and that $\Sigma^g \cup \Sigma \subseteq \Sigma^{g,h}$. This is again a simple reflection argument along the lines of the proof of Claim 2, and so again, we omit it. Being guided by a self-justifying system, Σ^h condenses well. □

We can now show that the pullback strategy $i(\Sigma)^i$ used in the proof of Lemma 50 is more reasonable. At the same time, we show that $L(\mathbb{R})$ of $\mathrm{Ult}(V, G)$ is essentially the same as $L(\mathbb{R})$ of $V[G][g]$, to a level strictly past β.

Lemma 56. *Let $G \subset (P(\omega_1)/NS)^V$ be $V[g]$-generic and let $i\colon V \to M = Ult(V, G)$ the generic embedding. Then*

1. *β^G ends a gap in M of the same sort that it ends in $V[g]$,*
2. *$i(\Sigma)^i \restriction HC^M$ is guided by a self-justifying system which seals this gap in M, and in turn determines the self-justifying system,*
3. *$W^*_{\beta^G+\omega}$ holds in M,*
4. *there is a fully elementary*
$$\sigma\colon J_{\beta^G+\omega}(\mathbb{R})^M \to J_{\beta^G+\omega}(\mathbb{R})^{V[G][g]}$$
which is the identity on ordinals, and such that
5. *$\sigma(i(\Sigma)^i \restriction HC^M) = \Sigma^G \restriction HC^{V[G][g]}$, and*
6. *$W^*_{\beta^G+\omega}$ holds in $V[g][G]$.*

Proof. Let $\bar{z} = \tau_l$, where $l \in M$ and l is $Col(\omega, \omega_1^V)$-generic over N_τ, with $p_0 \subseteq l$. Let P be an $i(\Sigma)^i$-iterate of $N_\tau[l]$ with P countable in M. Working outside M, we can form an $\mathbb{R}^M$-genericity iteration
$$P = P_0 \to P_1 \to P_2 \dots . P_\infty,$$
since $i(\Sigma^i)$ is an $\omega_1 + 1$ strategy in the sense of M, so the individual genericity iterations terminate. Put for $x \in \mathbb{R}^M$,
$$x \in B_i \Leftrightarrow \exists k \exists \delta (x \in i_{0,k}(\sigma_{A_i,\delta})_{l \star g_k}),$$
where $i_{0,k}\colon N_\tau \to P_k$ is the iteration map, δ is a Woodin of N_τ, g_k is $Col(\omega, \delta)$ generic over $N_\tau[l]$, and the $\sigma_{A_i,\delta}$ are as in the proof of Lemma 53. One can show that the B_i yield a self-justifying system which codes up truth at the end of a gap $[\alpha^M, \beta^M]$ in M. This is because there is enough about the A_i recorded in the first order theory of N_τ and the $\sigma_{A_i,\delta}$. See [21] for a similar argument. It follows from the same argument that the B_i are independent of the particular P or $\mathbb{R}^M$-genericity iteration chosen, and that $i(\Sigma)^i$ respects the B_i, in

that iterations by it move the $\tau^{N_\tau[l]}_{B_i,\delta}$ (derived from the $\sigma_{A_i,\delta}$) correctly. But the ordinals definable from $\sigma_{A_i,\delta}$ as i varies are cofinal in δ, and thus the B_i guide the iteration strategy $i(\Sigma)^i$, that is, they determine it. By 52, we have that M satisfies that HC is closed under the $P_n^{i(\Sigma)^i}$-sharp operator, for all n. From the paragraph above, $i(\Sigma)^i \restriction \mathrm{HC}^M$ is projectively equivalent to $\langle B_i \mid i < \omega\rangle$. So we get $W^*_{\beta^M+1}$ in M. Working in M with the first ω^2 of these operators, instead of just the first ω, we get $W^*_{\beta^M+\omega}$ in M. We refer the reader to [14] for this routine extension. We have now proved (1) through (3), except we don't yet know $\beta^M = \beta^G$. In particular, we have $W^*_{\beta^M+\omega}$ in M, and an elementary $\sigma\colon J_{\beta^M+\omega}(\mathbb{R})^M \to J_{\beta^M+\omega}(\mathbb{R})^{V[G][g]}$, because g is Cohen generic over $V[G]$. Now note that

$$i(\Sigma)^i \restriction \mathrm{HC}^M \in J_{\beta^M+\omega}(\mathbb{R})^M.$$

Also, we used nothing about $\bar{z}$ except that l is generic over N_τ and $p_0 \subseteq l$. In M, any such l determines a variant self-justifying system which also guides $i(\Sigma)^i \restriction \mathrm{HC}$. Letting C be the set of such l, $g \in \sigma(C)$, and thus

$$\langle A_i^G \mid i < \omega\rangle \text{ guides } \sigma(i(\Sigma)^i \restriction \mathrm{HC}^M),$$

and thus

$$\sigma(i(\Sigma)^i \restriction \mathrm{HC}^M) = \Sigma^G \restriction \mathrm{HC}^{V[G][g]}.$$

This gives us (1) through (5) in full, and of course, (6) is an immediate consequence. □

Remark 57. We eventually get $\beta^G = \beta$, but only after we have shown W^*_γ holds in $V[g]$. That is because the Foreman-Magidor argument requires a universally Baire prewellorder of length β in $V[g]$.

What we need from here on is just the following immediate consequence of Lemma 56:

Corollary 58. *Let G be NS^V-generic over $V[g]$, then*

$$V[g][G] \models \forall A \in H(\omega_1)\forall n(P_n^{\Sigma^G}(A)^\sharp \text{ exists and is } \omega_1\text{-iterable }).$$

Now let h be $Col(\omega, \omega_2^V)$-generic over $V[g]$, and $G \in V[g][h]$ be NS-generic over $V[g]$. Note that the extension from $V[g]G$ to $V[g][h]$ is by a partial order which, in $V[g]G$, is of size ω_1 and collapses ω_1. So $V[g]g$-to-$V[g][h]$ is a homogeneous extension. We shall show the mice $P_n^{\Sigma^G}(A)^\sharp$ given by 58 are definable from A in $V[g][h]$, thus in $V[g]$ when $A \in V[g]$. Definability comes from lifting their strategies to $V[g][h]$, and that comes from lifting the operators themselves to $V[g][[h]$. To do that, we need to use WRP in V, so we must consider the P_n^Σ-sharp operators on $H(\omega_3)^V$. The following lemma does the job.

Lemma 59. *For all $n < \omega$,*

(1) $V \models$ *for all transitive* $A \in H_{\omega_3}$ *such that* $N_\tau \in A$, $P_n^\Sigma(A)^\sharp$ *exists and is* ω_3*-iterable,*
(2) *if* $A \in H_{\omega_3}^V$ *and* P *is such that* $V \models$ *"*$P = P_n^\Sigma(A)^\sharp$ *is* ω_3*-iterable", then* $V[g][h] \models$ *"*$P = P_n^{\Sigma^h}(A)^\sharp$ *is* ω_1*-iterable", and*
(3) $V[g][h] \models$ *for all countable transitive* A *such that* $N \in A$, $P_n^{\Sigma^h}(A)$ *exists and is* ω_1*-iterable.*

Proof. By induction on n. We have already proved (1) when $n = 0$. Part (2) is trivial in this case, since the iterations of P are all linear iterations by its unique extender, and hence are all in V. For part (3), Note that the P_0^Σ-sharp operator determines itself on $V[g][h]$. More precisely, the P_0^Σ-sharp operator on $H(\omega_3)^V$ determines the $P_0^{\Sigma^h}$-sharp operator on $H(\omega_1)^{V[g][h]}$. For let A be countable transitive in $V[g][h]$, and say $A = \rho_{g\times h}$. Let $B \in V$ be the transitive closure of $\{\rho, N_\tau\}$. We have an ω_3^V-iterable $P = P_0^\Sigma(B)^\sharp$ in V. But then $P[g \times h]$ exists in $V[g][h]$, and we can obtain $P_0^{\Sigma^h}(A)^\sharp$ from it. This is because the determination of Σ^h from Σ we gave (via $\mathbb{R}$-genericity iterations which create a self-justifying system guiding Σ^h) is sufficiently local that if $M \models$ ZFC and $\Sigma \cap M \in M$ and $g, h \in M$, then $\sigma^h \cap M \in M$ and is uniformly-in-M definable over M from $\Sigma \cap M, g$, and h. Iterations of $P_0^{\Sigma^h}(A)^\sharp$ correspond to iterations of P as in (2). The latter stretch Σ into Σ, so the former stretch Σ^h into Σ^h.

Now suppose (1)–(3) hold for $n = k$. We consider (1) for $n = k+1$. We first consider the case $A \in H_(\omega_2)^V$. In $V[g]$, let B be the first admissible set over $\{A, g\}$, so that $N \in B$. By Lemma 58 we have P in $V[g][G]$ such that

$$V[g][G] \models P = P_{k+1}^{\Sigma^G}(B)^\sharp,$$

in the sense that P has the first order properties, and is $\omega_1^{V[g][G]}$-iterable via a strategy which moves Σ^G to itself. We claim that

$$V[g][h] \models P = P_{k+1}^{\Sigma^h}(B)^\sharp,$$

in the sense that P is $\omega_1^{V[g][h]}$ iterable in $V[g][h]$ via a strategy which moves Σ^h to itself. The iteration strategy for P in $V[g][h]$ is the one guided by the Q-structures provided by (3) for $n = k$. Let Γ be this strategy, and suppose Γ fails in $V[g][h]$. Let $\mathcal{O}^h$ be the $P_k^{\Sigma^h}$-sharp operator of $V[g][h]$, and let $\mathcal{O} = \mathcal{O}^h \restriction V[g][G]$. So $\mathcal{O}$ is defined on $A \in H(\omega_2)^{V[g][G]}$ with $N \in A$. We have that $\mathcal{O} \in V[g][G]$, because the extension to $V[g][h]$ is homogeneous, and $\mathcal{O}^h$ is definable in $V[g][h]$ from Σ. Moreover, $\mathcal{O}$ determines the full $\mathcal{O}^h$ in $V[g][h]$ via the process we have described. So Γ is definable in $V[g][h]$ from $\mathcal{O}$, and

$$V[g][G] \models \text{“it is forced that the strategy for } P \text{ determined by } \mathcal{O} \text{ fails”}.$$

From the point of view of $V[g][G]$, the forcing in question is just $Col(\omega, \omega_1)$. But now, working in $V[g][G]$, let

$$\pi \colon S \to V_\eta,$$

where S is countable transitive, with everything relevant in its range. Let l be S-generic for the collapse of ω_1^S, with $l \in V[g][G]$. It is then easy to see that $\pi^{-1}(\mathcal{O})$ is contained in the $P_k^{\Sigma^G}$-sharp operator of $V[g][G]$, and what it determines on $S[l]$ is also contained in the $P_k^{\Sigma^G}$-sharp operator of $V[g][G]$. Since P did have a strategy in $V[g][G]$ guided by this operator, we have a contradiction, proving our claim.

Now we can invert the extension leading from A to B, getting a Σ^h premouse Q over A such that

$$P = \text{ canonical re-arrangement of } Q[g] \text{ as a premouse over } B.$$

By the homogeneity of the forcing and the definablity of P in $V[g][h]$, we get that inductively that all levels of Q are in V, and that all trees to which Σ^h is applied in such levels are in V. Thus Q is a Σ-premouse in V. The iteration strategy for P in $V[g][h]$ induces a strategy for Q in V, and this strategy is in V by homogeneity, and it witnesses

$$Q = P_{k+1}^{\Sigma}(A)^\sharp$$

in V, and that Q is ω_3-iterable in V. We now use WRP to extend the P_{k+1}^{Σ}-sharp operator to $H(\omega_3)$ in V, just as we did in the $n = 0$ case. For $A \subseteq \omega_2$, the key definition is

$$(\phi, \vec{\alpha}) \in P_{k+1}^{\Sigma}(A)^\sharp \quad \Leftrightarrow \quad \text{for club many } \sigma \in P_{\omega_1}(H_{\omega_2}) \quad (\phi, \vec{\alpha}_\sigma) \in P_{k+1}^{\Sigma^{\pi_\sigma}}(A_\sigma)^\sharp.$$

Here we use Lemma **??** to see that for each such $\sigma \in P_{\omega_1}(\omega_2)$, $P_{k+1}^{\Sigma^{\pi_\sigma}}(A_\sigma)^\sharp$ exists, and is ω_3-iterable. Just as in the $n = 0$ case, we get that everything is decided on a club, so that the definition yields a structure with the first order properties of $P_{k+1}^{\Sigma}(A)^\sharp$. An argument parallel to that in the $n = 0$ case shows that this structure interprets $\dot{\Sigma}$ as Σ, and that it is ω_3-iterable in a way that moves Σ to itself. Here one uses the corresponding properties of the $P_{k+1}^{\Sigma^{\pi_\sigma}}(A_\sigma)^\sharp$, and the fact that Σ collapses into its pullbacks under Skolem hulls. This finishes the proof of (1) at $k+1$. We leave the straightforward proofs of (2) and (3) at $k+1$ to the reader. $\square$

We have finally done what we set out to do in this section.

Lemma 60. *The following holds in $V[g]$. For all transitive $A \in H(\omega_1)$ such that $N \in A$, and all $n < \omega$, $P_n^{\Sigma^g}(A)^\sharp$ exists and is ω_1-iterable. Hence $W^*_{\gamma+1}$ holds in $V[g]$.*

Proof. This follows at once from (3) of 59, and the homogeneity of $Col(\omega, \omega_1)$. Every set in $J_\gamma(\mathbb{R})^{V[g]}$ is (boldface) projective in Σ^g. So the $P_n^{\Sigma^g}(A)^\sharp$ are the desired coarse capturing mice. □

Lemma 61. *$I_{\gamma+1}$ holds.*

Proof. We have shown that the $P_n^{\Sigma^G}$-sharp operator of $V[g][G]$, when restricted to $\mathrm{HC}^{V[g]}$, is just the $P_n^{\Sigma^g}$-sharp operator of $V[g]$. Projective-in-Σ truth is coded into these operators, so we get

$$(\mathrm{HC}^{V[g]}, \in, \Sigma^g) \prec (\mathrm{HC}^{V[g][G]}, \in, \Sigma^G).$$

But Σ^g codes truth at the bottom of the Levy hierarchy over $J_\gamma(\mathbb{R})^{V[g]}$, and Σ^G codes truth at the bottom of the Levy hierarchy over $J_{\gamma^G}(\mathbb{R})^{V[g][G]}$, where $\gamma^G = \beta^G$ if our gap was weak, and $\gamma^G = \beta^G + 1$ otherwise. (Truth is coded via $\mathbb{R}$-genericity iterations which determine self-justifying systems at the end of these gaps, as in our argument.) So we get from the line displayed above an embedding

$$\pi \colon J_{\gamma+1}(\mathbb{R})^{V[g]} \to J_{\gamma^G+1}(\mathbb{R})^{V[g][G]}$$

which is Σ_1 elementary. But in $V[g]$ we have an ω_1 Universally Baire prewellorder of length γ, so we can use the Foreman-Magidor argument to show $\gamma = \gamma^G$, and $\pi =$ identity. □

Repeating the relevant arguments ω times gives $W^*_{\gamma+\omega}$ in $V[g]$ as well as $I_{\gamma+\omega}$.

6. Concluding Remarks

Like many of the well-known consequences of $\mathsf{MM}(c)$, our hypotheses follow from the Strong Reflection Principle of Todorcevic, denoted $\mathsf{SRP}(\omega_2)$, which asserts that for every projective stationary subset S of $[\omega_2]^\omega$ there is an ordinal $\delta < \omega_2$ so that $S \cap [\delta]^\omega$ contains a club in $[\delta]^\omega$. Thus, our main theorem gives $\mathsf{AD}^{L(\mathbb{R})}$ from $\mathsf{SRP}(\omega_2)$ as well. While this represents the best known lower bound for the strength of $\mathsf{SRP}(\omega_2)$, and even $\mathsf{MM}(c)$, these principles are almost certainly much stronger.[5] Moreover, in a precise sense, our arguments can not take us much farther. In section 9.5 of [22] Woodin defines principles $\mathsf{SRP}^*(\omega_2)$

[5]They can be obtained via forcing from a supercompact cardinal (see [4]) or from $\mathsf{AD}_\mathbb{R} + \Theta$ regular (see 10.88 of [22]).

and $\mathsf{WRP}^*_{(2)}(\omega_2)$ and shows that the latter is a consequence of the former if NS is saturated (see Lemma 9.93 of [22]). $\mathsf{SRP}^*(\omega_2)$ asserts the existence of a normal fine ideal $I \subset P([\omega_2]^\omega)$ with the following two properties: (1) For every $T \in P(\omega_1) \setminus NS$ the set

$$S_T = \{\sigma \in [\omega_2]^\omega \mid \sigma \cap \omega_1 \in T\}$$

is I-positive, and (2) for every $S \subset [\omega_2]^\omega$ which satisfies $S \cap S_T \notin I$ for every $T \in P(\omega_1) \setminus NS$, there is $\gamma < \omega_2$ such that $S \cap [\gamma]^\omega$ contains a club in $[\gamma]^\omega$. $\mathsf{WRP}^*_{(2)}(\omega_2)$ asserts the existence of a normal fine ideal I with property (1) above so that any pair $S, T \notin I$ simultaneously reflect to stationary sets in some $[\gamma]^\omega$. Woodin obtains $SRP^*(\omega_2)$ together with the saturation of NS in a $\mathbb{P}_{max}$ extension of a determinacy model whose existence is equiconsistent with ω^2 Woodin cardinals.

Theorem 62. *(Woodin; 9.102 of* [22]*) The following are equiconsistent.*

(1) *$F \cap L(F, \mathbb{R})$ is an ultrafilter where F is the club filter on $[\mathbb{R}]^\omega$ and AD holds in the model $L(F, \mathbb{R})$*
(2) *There exists a set of Woodin cardinals of order type ω^2*

Moreover, if G is $\mathbb{P}_{max}$ generic over $L(F, \mathbb{R})$ as in (1) then

$$L(F, \mathbb{R})[G] \models \mathsf{SRP}^*(\omega_2) \text{ and NS saturated .}$$

Woodin remarks in 9.98 of [22] that his proof of Theorem 10 also proves PD from NS saturated and $\mathsf{WRP}^*_{(2)}(\omega_2)$. The same is true of our argument.

Corollary 63. *Assume NS saturated, $\mathsf{WRP}^*_{(2)}(\omega_2)$ holds, and $2^\omega \leq \omega_2$. Then $L(\mathbb{R}) \models AD$.*

Proof. This amounts to checking that $\mathsf{WRP}^*_{(2)}(\omega_2)$ can serve in the place of $WRP_{(2)}(\omega_2)$ in every $H(\omega_2)$ to $H(\omega_3)$ lifting argument. For example in Lemma 14 we would show that the sets S_t are measured by the filter dual to I (as opposed to the club filter). One therefore gets $2^{\omega_1} = \omega_2$ as in Lemma 8 and the rest of the proof is the same as in the proof of the Main Theorem. □

Very likely the $2^\omega \leq \omega_2$ hypothesis can be dropped, although we haven't thought this through.[6] Thus the consistency strength of NS saturated together with $\mathsf{WRP}^*_{(2)}(\omega_2)$ and $2^\omega \leq \omega_2$ is somewhere in the interval

$$(\omega \text{ Woodins, } \omega^2 \text{ Woodins}]$$

and we have reason to believe that the following conjecture is true.

[6]The point is that we use $2^{\omega_1} = \omega_2$ at various points and the hypotheses NS saturated and $\mathsf{WRP}^*_{(2)}(\omega_2)$ give all of Lemma 8 except Todorcevic's bound on 2^ω.

Conjecture 64. The following are equiconsistent.

(1) There exists a set of Woodin cardinals of order type ω^2
(2) NS is saturated and $\mathsf{WRP}^*_{(2)}(\omega_2)$ holds.
(3) NS is saturated and $\mathsf{SRP}^*(\omega_2)$ holds.

The first step is to prove that $K(\mathbb{R}) \models \mathsf{AD}$. We leave this for another time.

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