The length-ω_1 open game quantifier propagates scales

John R. Steel

June 5, 2006

0 Introduction

We shall call a set T an ω_1-*tree* if and only if

$$T \subseteq \bigcup_{\alpha<\omega_1} \omega^\alpha,$$

and whenever $p \in T$, then $p \restriction \beta \in T$ for all $\beta < \mathrm{dom}(p)$. (This diverges a bit from standard usage.) The set of ω_1-branches of T is

$$[T] = \{f \colon \omega_1 \to \omega \mid \forall \alpha < \omega_1 (f \restriction \alpha \in T)\}.$$

$[T]$ is just a typical closed set in the topology on ω^{ω_1} whose basic neighborhoods are the sets of the form $N_p = \{f \colon \omega_1 \to \omega \mid p \subseteq f\}$, where $p \colon \alpha \to \omega$ for some countable α. Associated to T is a closed game $\mathcal{G}(T)$ on ω of length ω_1: at round α in $\mathcal{G}(T)$, player I plays m_α, then player II plays n_α. Letting $f(\alpha) = \langle m_\alpha, n_\alpha \rangle$ for all $\alpha < \omega_1$, we say that player II wins the run of $\mathcal{G}(T)$ determined by f iff $f \in [T]$. (We code sequences from ω by prime powers, so that $\langle n_0, ..., n_k \rangle = \prod_{i \leq k} p_i^{n_i+1}$, and the decoding is given by $(n)_i =$ (exponent of p_i in n)-1, where p_i is the i^{th} prime.)

Of course, not all such closed games are determined; indeed, every game of length ω can be regarded as a clopen game of length ω_1. However, the determinacy of $\mathcal{G}(T)$ for definable T does follow from a large cardinal hypothesis, in virtue of the following beautiful theorem of Itay Neeman:

Theorem 0.1 (Neeman, [1],[2]) *Suppose that for any real x, there is a countable, ω_1+1-iterable mouse M such that $x \in M$ and $M \models \mathsf{ZFC}^- +$ "there is a measurable Woodin cardinal"; then $\mathcal{G}(T)$ is determined whenever T is a ω_1-tree such that T is definable over $(H_{\omega_1}, \in)$ from parameters.*

The definability restriction on T is equivalent to requiring that T be coded by a projective set of reals. For that reason, we shall call the conclusion of Neeman's theorem ω_1-*open-projective determinacy*. Neeman's proof works, under a natural strengthening of its large cardinal hypothesis, whenever T is coded by a Hom_∞ set of reals. In this form, Neeman's result obtains as much determinacy from large-cardinal-like hypotheses as has been proved to date.[1]

Associated to any game we have a game quantifier:

Definition 0.2 *Let T be an ω_1-tree; then*

$$\Game^{\mathrm{I}}(T) = \{p \in \omega^\omega \mid p \textit{ is a winning position for } \mathrm{I} \textit{ in } \mathcal{G}(T)\},$$

and

$$\Game^{\mathrm{II}}(T) = \{p \in \omega^\omega \mid p \textit{ is a winning position for } \mathrm{II} \textit{ in } \mathcal{G}(T)\}.$$

Definition 0.3

$$\Game^{\omega_1}(\textit{open-analytical}) = \{\Game^{\mathrm{I}}(T) \mid T \textit{ is an } \omega_1\textit{-tree which is definable over } (H_{\omega_1}, \in)\},$$

and

$$\Game^{\omega_1}(\textit{closed-analytical}) = \{\Game^{\mathrm{II}}(T) \mid T \textit{ is an } \omega_1\textit{-tree which is definable over } (H_{\omega_1}, \in)\}.$$

Clearly, the determinacy of the games in question implies that the pointclasses $\Game^{\omega_1}$(open-analytical) and $\Game^{\omega_1}$(closed-analytical) are duals. The following observations relate these pointclasses to more familiar ones.

Proposition 0.4 *For any $A \subseteq \omega^\omega$, the following are equivalent:*

(1) A is $\Game^{\omega_1}$(closed-analytical),

(2) there is a Σ^2_1 formula $\theta(v)$ such that for all $x \in \omega^\omega$,

$$x \in A \textit{ iff } \overset{\mathrm{Col}(\omega_1,\mathbb{R})}{\Vdash} \theta(\check{x}),$$

where $\mathrm{Col}(\omega_1, \mathbb{R})$ is the partial order for adding a map from ω_1 onto $\mathbb{R}$ with countable conditions.

[1]Literally speaking, Neeman's theorem uses a mouse existence hypothesis, rather than a large cardinal hypothesis. It is not known how to derive this mouse existence hypothesis from any true large cardinal hypothesis, because it is not known how to prove the required iterability. Presumably, if there is a measurable Woodin cardinal, then there are mice of the sort needed in Neeman's theorem.

Proposition 0.5 *If* CH *holds, then* $\Game^{\omega_1}$(*closed-analytical*) $= \Sigma^2_1$.

Corollary 0.6 *If* CH *and* ω_1*-open-analytical determinacy hold, then* $\Game^{\omega_1}$(*open-analytical*) $= \Pi^2_1$.

We shall omit the easy proofs of these results.

It is natural to ask whether ω_1-open-projective determinacy implies the pointclasses $\Game^{\omega_1}$(open-analytical) and $\Game^{\omega_1}$(closed-analytical) are well-behaved from the point of view of descriptive set theory. In this note we shall extend Moschovakis' periodicity theorems to this context, and show thereby that ω_1-open-projective determinacy implies that $\Game^{\omega_1}$(open-analytical) has the Scale Property, and that there are canonical winning strategies for the ω_1-open-projective games won by player I. By 0.6, adding CH to our hypotheses gives the Scale Property for Π^2_1.

Our arguments are quite close to those of sections 1 and 2 of [3], which prove the same results for the quantifiers associated to certain *clopen* games of length ω_1. Therefore, in this paper we shall only describe the simple changes to [3] which are needed for the full theorems on open games, and refer the reader to [3] for the long stretches of the proofs which agree in every detail.

1 The Prewellordering Property

Moschovakis' argument easily yields the prewellordering property for $\Game^{\omega_1}$(open-analytical).

Theorem 1.1 *If* ω_1*-open-projective determinacy holds, then the pointclass* $\Game^{\omega_1}$(*open-analytical*) *has the prewellordering property.*

Proof. Let T be an ω_1 tree which is definable over $(H_{\omega_1}, \in)$, and let

$$B = \Game^{\mathrm{I}}(T).$$

We define a prewellordering on B by means of a game $G(p, q)$ which compares the values for I of positions $p, q \in B$. In $G(p, q)$ there are two players, F and S, and two boards, the p-board and the q-board. The play takes place in rounds, the first being round ω. In round $\alpha \geq \omega$ of $G(p, q)$:

(a) F plays as I in round α of $\mathcal{G}(T)$ on the q-board, then

(b) S plays as I in round α of $\mathcal{G}(T)$ on the p-board, then

(c) F plays as II in round α of $\mathcal{G}(T)$ on the p-board, then

(d) S plays as II in round α of $\mathcal{G}(T)$ on the q-board.

This play produces $f \supseteq p$ on the p-board and $g \supseteq q$ on the q-board, with $f, g \in \omega^{\omega_1}$. Player S wins this run of $\mathcal{G}(p,q)$ iff for some $\alpha < \omega_1$, $f \upharpoonright \alpha \notin T$ but for all $\beta < \alpha, g \upharpoonright \beta \in T$. In other words, S must win $\mathcal{G}(T)$ as I on the p-board, and not strictly after F wins as I (if he does) on the q-board. Put

$$p \leq^* q \Leftrightarrow p, q \in A \text{ and } S \text{ has a winning strategy in } G(p,q).$$

Let $\mathcal{G}^0(p,q)$ be the same as $\mathcal{G}(p,q)$, except that a run (f,g) such that neither player has won as I (i.e. such that $f \in [T]$ and $g \in [T]$) is a win for S, rather than F.

Lemma 1.2 *Let $p_0 \in \Game^{\mathrm{I}}(T)$, and suppose that for all $n \geq 0$, Σ_n is either a winning strategy for F in $G(p_n, p_{n+1})$, or a winning strategy for S in $G^0(p_{n+1}, p_n)$; then only finitely many Σ_n are for F.*

Proof. Let $\langle p_n \mid n < \omega \rangle$ and $\langle \Sigma_n \mid n < \omega \rangle$ be as in the hypothesis. Let τ be winning for I in $\mathcal{G}(T)$ from p_0.

Note that in any case, Σ_n plays for II on the p_n board, and for I on the p_{n+1} board. Playing τ and the Σ_n's together in the standard game diagram, we get $\langle u_n \mid n < \omega \rangle$ such that for all n,

(1) u_n is a run according to Σ_n of the appropriate game ($G(p_n, p_{n+1})$ if Σ_n is for F, and $G^0_(p_{n+1}, p_n)$ if Σ_n is for S), and

(2) u_n and u_{n+1} have a common play r_{n+1} on their p_{n+1}-boards,

(3) the play r_0 on the p_0-board of u_0 is according to τ.

As usual, $u_n(\alpha)$ is determined by induction on α, simultaneously for all n.

Because τ was winning for I, we have $r_0 \notin [T]$. It follows by induction that $r_n \notin [T]$ for all n. Let α_n be least such that $r_n \upharpoonright \alpha_n \notin [T]$. Since Σ_n won its game, we have that $\alpha_{n+1} \leq \alpha_n$ if Σ_n is for S, and $\alpha_{n+1} < \alpha_n$ if Σ_n is for F. Thus only finitely many Σ_n are for F. $\square$

Corollary 1.3 *$\leq^*$ is a prewellorder of B.*

Proof.

(1) *Reflexive:* if $\neg p \leq^* p$, then $\langle p, p, p, ... \rangle$ violates 1.2.

(2) *Transitive:* if $p \leq^* q \leq^* r$, but $\neg p \leq^* r$, then $\langle r, q, p, r, q, p, r, ... \rangle$ violates 1.2.

(3) *Connected:* otherwise $\langle p, q, p, q, ...\rangle$ violates 1.2.

(4) *Wellfounded:* clear from 1.2.

$\square$

Corollary 1.4 *If $q \in B$, then for any p, $p \leq^* q \Leftrightarrow$ II has a winning strategy in $\mathcal{G}^0(p,q)$.*

Proof. Otherwise, let $p_n = q$ if n is even, and $p_n = p$ if n is odd. Let Σ_n be winning for S in $G^0(p,q)$ if n is even, and let Σ_n be winning for F in $G(p,q)$ if n is odd. We obtain a contradiction to 1.2. $\square$

So if $q \in B$, then we have

$$\begin{aligned} p \in B \wedge p \leq^* q \quad &\Leftrightarrow \quad S \text{ has a winning strategy in } \mathcal{G}(p,q) \\ &\Leftrightarrow \quad S \text{ has a winning strategy in } \mathcal{G}^0(p,q). \end{aligned}$$

The first equivalence shows $\{p \mid p \leq^* q\}$ is uniformly $⅁^{\omega_1}$(open-analytical) in q, the second shows that it is uniformly $⅁^{\omega_1}$(closed-analytical) in q. Thus $\leq^*$ determines a $⅁^{\omega_1}$(open-analytical) norm on B.

This completes the proof of 1.1. $\square$

2 The Scale Property

In order to prove the scale property for $⅁^{\omega_1}$(open-analytical), we need to add to the comparison games of 1.1 additional moves. As in [3], these additional moves reflect the value which the players assign to one-move variants of positions they reach during a run of the game. One needs that such positions are canonically coded by reals in order to make sense of this. For this reason, in [3] we demanded that there be a function F assigning to possible positions $p \in T$ a wellorder $F(p) \in \mathrm{WO}$ such that $F(p)$ has order type $\mathrm{dom}(p)$. What is new here is just the realization that one can relax this a bit, by demanding only that our game tree T incorporates the rule that player I must provide a code of $\mathrm{dom}(p)$ as soon as p has been reached, before II is required to make any further moves.

Theorem 2.1 *If ω_1-open-projective determinacy holds, then the pointclass $⅁^{\omega_1}$(open-analytical) has the scale property.*

Proof. Let $\hat{T}$ be an ω_1 tree which is definable over $(H_{\omega_1}, \in)$, and let $B = ⅁^{\mathrm{I}}(\hat{T})$. We assume without loss of generality that in $\mathcal{G}(\hat{T})$, II plays only from $\{0,1\}$. We may as well also assume that I and II only move in $\mathcal{G}(\hat{T})$ at finite rounds, or at rounds of the form $\omega\eta + \omega$ where

$\eta \geq 1$. Finally, we assume that for $\eta \geq 1$, player I must produce a code of $\omega\eta + \omega$ in rounds $\omega\eta + 1$ through $\omega\eta + \omega$. Formally, for any $r \in \omega^{<\omega_1}$, let

$$q_r(n) = r(n) \text{ if } \eta < \omega$$

and

$$q_r(\omega + \eta) = r(\omega + \omega\eta + \omega)$$

for all $\eta \geq 0$ such that $\omega + \omega\eta + \omega \in \mathrm{dom}(r)$. Set

$$w_r^\eta(n) = r(\omega\eta + n + 1)_0,$$

for η such that $\omega + \omega\eta + \omega \leq \mathrm{dom}(r)$. For $r \in \omega^{<\omega_1}$, we put $r \in A$ iff

(a) $\mathrm{dom}(r) = \omega + \omega\xi + \omega$, for some ξ,

(b) $\forall \eta \in \mathrm{dom}(q_r)(q_r \upharpoonright \eta \in \hat{T}$, but $q_r \notin \hat{T})$, and

(c) if $\omega\eta + \omega \leq \mathrm{dom}(r)$, then $w_r^\eta \in \mathrm{WO}^*$ and $|w_r^\eta| = \omega\eta + \omega$.

Here WO^* is the set of reals x coding wellorders of ω of length $|x| > \omega$, and such that $|0|_x = \omega$. (This last very technical stipulation simplifies something later.) For $r \in \omega^{<\omega_1}$, put

$$r \in T \Leftrightarrow \forall \xi \leq \mathrm{dom}(r)(r \upharpoonright \xi \notin A).$$

Thus in $\mathcal{G}(T)$, player I is just trying to reach a position in A.

Claim 1. $B = \Game^{\mathrm{I}}(T)$.

Proof. This is clear. It is worth noting that we use the Axiom of Choice at this point, however. □

As we said, our only new idea here is to work with T, rather than $\hat{T}$, in defining the comparison games producing a scale on B.

We shall use the notation of the proof of Theorem 1 of [3] as much as possible. The counterpart of F there is given by: $\mathrm{dom}(F)$ consists of those $r \in \omega^{<\omega_1}$ which satisfy (a) and (c) above, and for such r

$$F(r) = w_r^\eta, \text{ where } \mathrm{dom}(r) = \omega\eta + \omega.$$

$$r^* = \langle F(r), x \rangle,$$

where $x(n) = r(|n|_{F(r)})$. For $D \subseteq \mathrm{dom}(F)$, we let $D^* = \{r^* \mid r \in D\}$. Let

$$\begin{aligned} C(i,j,k,y) \quad \Leftrightarrow \quad & \exists r \in \mathrm{dom}(F)(y = r^* \wedge |i|_{F(r)} = |j|_{F(r \upharpoonright \alpha)} \\ & \text{where } |k|_{F(r)} = \alpha = \omega\xi + \omega, \text{ for some } \xi. \end{aligned}$$

Note that C and F are Δ^1_2. Let $\vec{\rho}$ be a Δ^1_2 scale on C, and let $\vec{\sigma}$ be an analytical scale on T^*. Let θ^k_i, for $0 \leq k \leq 4$ and $i < \omega$, be the norms on A^* defined from $\vec{\rho}$ and $\vec{\sigma}$ exactly as in [3, p. 58]. Let

$$D(k, y) \quad \Leftrightarrow \quad \exists r \in \mathrm{dom}(F)(y = r^* \wedge \exists \xi \geq 1(|k|_{F(r)} = \omega\xi + \omega)),$$

let $\vec{\nu}^0, \vec{\nu}^1$ be an analytical scales on D and $\neg D$, and set

$$\theta^5_{\langle k,i \rangle}(y) = \begin{cases} \nu^0_i(k, y), & \text{when } D(k, y), \\ \nu^1_i(k, y), & \text{when } \neg D(k, y). \end{cases}$$

Let $\vec{\theta}^6$ be a very good scale on A^*, and set, for $p^* \in A^*$,

$$\psi_i(p^*) = \langle \mathrm{dom}(p), \theta^0_0(p^*), ..., \theta^6_0(p^*), ..., \theta^0_i(p^*), ..., \theta^6_i(p^*) \rangle,$$

or rather, let $\psi_i(p^*)$ be the ordinal of this tuple in the lexicographic order. Just as in [3], the norms ψ_i determine the values we assign to runs of the comparison games which yield a scale on B.

Let $p, q \in \Game^{\mathrm{I}}(T)$, and let $k \in \omega$. We shall define a game $G_k(p, q)$. The players in $G_k(p, q)$ are F and S. They play on two boards, the p board and the q board, and make additional moves lying on neither board. On the p board, S plays $\mathcal{G}(T)$ from p as I, while F plays as II. On the q board, F plays $\mathcal{G}(T)$ from q as I, while S plays as II. Play is divided into rounds, the first being round ω. We now describe the typical round.

We shall require that the moves in $G_k(p, q)$ on the p and q boards which are meaningless for $\mathcal{G}(T)$ be 0. So in round α of $G_k(p, q)$, for $\alpha = \omega$ or α a limit of limit ordinals, F and S must each play 0's on both boards. We make the technical stipulation that F always proposes k at round ω. (This will make it more convenient to describe the payoff condition of $G_k(p, q)$.) No further moves are made in round α, for $\alpha = \omega$ or α a limit of limits.

Round α, for α a successor ordinal:

(a) F plays as I in round α of $\mathcal{G}(T)$ on the q-board, then

(b) S plays as I in round α of $\mathcal{G}(T)$ on the p-board, then

(c) F, S play 0 as II in round α of $\mathcal{G}(T)$ on the p, q-boards, respectively.

Round α, when $\exists \xi \geq 1(\alpha = \omega\xi + \omega)$:

(a) F makes I's α-th move on the q board, then S makes I's α-th move on the p-board.

(b) F now proposes some i such that $0 \leq i \leq k$ and $(i)_0 \in \{0, 1\}$.

(c) S either accepts i, or proposes some j such that $0 \leq j < i$ and $(j)_0 \in \{0, 1\}$.

(d) Let $t \leq k$ be the least proposal made during (b) and (c):

Case 1. $t \neq 0$. Then F and S must play $(t)_0$ as II's $\omega + \alpha$-th move on the p and q boards respectively.

Case 2. $t = 0$. Then F plays any $m \in \{0, 1\}$ as II's $\omega + \alpha$-th move on the p board, after which S plays any $n \in \{0, 1\}$ as II's $\omega + \alpha$-th move on the q board.

This completes our description of round α.

Play in $G_k(p, q)$ continues until one of the two boards reaches a position in A. If this never happens, then F wins $G_k(p, q)$. If one of the two boards reaches a position in A strictly before the other board does, then the player playing as I on that board wins $G_k(p, q)$. We are left with the case that we have a run u of $G_k(p, q)$ and r, s with $p \subseteq r$ and $q \subseteq s$ the runs of $\mathcal{G}(T)$ on the two boards, and $r, s \in A$. Letting $\beta = \mathrm{dom}(r) = \mathrm{dom}(s)$, we must have $\beta = \omega\xi + \omega$ for some $\xi \geq 1$. Let $e \in \omega$ be the least n such that for some $\alpha \geq \omega$, $|(n)_0|_{F(r)}| = \alpha$, and $(n)_1$ was the least proposal made during round α. (Our technical stipulations guarantee that $\langle 0, k \rangle$ is such an n.) We call e the *critical number* of u, and write $e = \mathrm{crit}(u)$; thus $\mathrm{crit}(u) \leq \langle 0, k \rangle$. Let $\alpha = |(e)_0|_{F(r)}$. If F proposed $(e)_1$ during round α, then

$$S \text{ wins } u \text{ iff } \psi_e(r^*) \leq \psi_e(s^*).$$

If S proposed $(e)_1$ during round α, then

$$S \text{ wins } u \text{ iff } \psi_e(r^*) < \psi_e(s^*).$$

This completes our description of $G_k(p, q)$.

Let $G_k^0(p, q)$ be just like $G_k(p, q)$, except that if neither board reaches a position in A after ω_1 moves, then it is S who wins, rather than F.

Lemma 2.2 *Let $p_0 \in \Game^{\mathrm{I}}(T)$, and suppose that for all $n \geq 0$, Σ_n is either a ws for F in $G_k(p_n, p_{n+1})$, or a ws for S in $G^0(p_{n+1}, p_n)$; then only finitely many Σ_n are for F.*

We omit the proof of 2.2, as it is a direct transcription of the corresponding lemma in [3].

For $p, q \in \Game^{\mathrm{I}}(T)$, we put

$$p \leq_k q \text{ iff II has a winning strategy in } G_k(p, q).$$

Corollary 2.3 *(a) $\leq_k$ is a prewellorder of $\Game^{\mathrm{I}}(T)$.*

(b) For $q \in \Game^{\mathrm{I}}(T)$, S has a winning strategy in $G_k(p,q)$ iff S has a winning strategy in $G^0_k(p,q)$.

(c) $\leq_k$ determines a $\Game^{\omega_1}$(open-analytical) norm on $\Game^{\mathrm{I}}(T)$.

Proof. Just as in the proofs of 1.3 and 1.4. □

Now for $p \in \Game^{\mathrm{I}}(T)$, let

$$\phi_k(p) = \text{ ordinal of } p \text{ in } \leq_k.$$

Lemma 2.4 *$\vec{\phi}$ is a semiscale on $\Game^{\mathrm{I}}(T)$.*

Proof. The proof is very close to that of [3, Lemma 2], so we just indicate the small changes needed.

Let $p_n \to p$ as $n \to \infty$, with $p_n \in \Game^{\mathrm{I}}(T)$ for all n. Let τ be a winning strategy for I in $\mathcal{G}(T)$. Suppose $p_{n+1} \leq_n p_n$, as witnessed by the winning strategy Σ_n for S in $G_n(p_{n+1}, p_n)$, for all n. We must show that $p \in \Game^{\mathrm{I}}(T)$. Suppose toward contradiction that σ is a winning strategy for II in $\mathcal{G}(T)$.[2]

If $r_n \in \mathrm{dom}(F)$ for all n, then we write

$$r = \lim\nolimits^*_{n\to\infty} r_n$$

for the same notion of "convergence in the codes" as defined in [3].

We define by induction on rounds runs u_n of $G_n(p_{n+1}, p_n)$ according to Σ_n. We arrange that u_n and u_{n+1} agree on a common play r_{n+1} extending p_{n+1} on the p_{n+1} board, and that the play r_0 extending p_0 on the p_0 board is by τ. So assume that $u_n \restriction \alpha$ and $r_n \restriction \alpha$ are given for all n.

If $\alpha = \omega$ or α is a limit of limit ordinals, we set $r_n(\alpha) = \langle 0, 0\rangle$, and thus $u_n(\alpha) = \langle 0, 0, n, 0, 0\rangle$, for all n. None of these moves count for anything, of course.

If α is a successor ordinal, then only I's move in $\mathcal{G}(T)$ counts. Let

$$a_0 = \tau(r_0 \restriction (\alpha)),$$

and

$$a_{n+1} = \Sigma_n((u_n \restriction \alpha)^\frown\langle a_n\rangle)$$

fill in the α-th column of I's plays in our diagram, and set $r_n(\alpha) = \langle a_n, 0\rangle$, and thus $u_n(\alpha) = \langle a_n, a_{n+1}, 0, 0\rangle$, for all n.

[2] Unfortunately, there are some typos in the proof of Lemma 2 of [3] which confuse τ with σ at various points.

Finally, suppose $\alpha = \omega\xi + \omega$ for some $\xi \geq 1$. Again, let $a_0 = \tau(r_0 \restriction (\alpha))$, and $a_{n+1} = \Sigma_n((u_n \restriction \alpha)^\frown\langle a_n \rangle)$ for all n. If a_n is eventually equal to some fixed a, and if $\lim^*_{n\to\infty} r_n \restriction (\alpha) = r$, where r is a play by σ of length $\omega\xi + \omega$ for some $\xi \geq 1$, then we set

$$d = \sigma(r^\frown\langle a \rangle).$$

If not, then we set $d = 0$.

We proceed now exactly as in [3]. For any n, let i_n be the largest i such that $\langle d, i \rangle \leq n$ and $\Sigma_n((u_n \restriction \alpha)^\frown\langle a_n, \langle d, i \rangle\rangle) =$ "accept", if such an i exists. Let $i_n = 0$ otherwise.

Case 1. $i_n \to \infty$ as $n \to \infty$.

Pick n_0 such that $i_n > 0$ for $n \geq n_0$. The proposal pair in $u_n(\alpha)$ is $\langle b_n, c_n \rangle$, where $b_n = \langle d, i_n \rangle$ for $n \geq n_0$ and $b_n = 0 =$ freedom for $n < n_0$, and $c_n =$ accept for all n. Let $d_n = d$ if $n \geq n_0$, and

$$d_n = \Sigma_n((u_n \restriction \alpha)^\frown\langle a_n, a_{n+1}, b_n, c_n, d_{n+1} \rangle)$$

if $n < n_0$.

Case 2. Otherwise.

Again, we define the $u_n(\alpha)$ exactly as in [3]. We shall not repeat the definition here in this case.

This completes the definition of the u_n and r_n. Since r_0 is a play by τ, $r_0 \restriction \alpha \in A$ for some α. Letting α_0 be the least α such that $r_n \restriction \alpha \in A$ for some n, we have $r_n \restriction \alpha_0 \in A$ for all sufficiently large n because the Σ_n's won for S. To save notation, let us assume $r_n \restriction \alpha_0 \in A$ for all n. Note that $\alpha_0 = \omega\xi + \omega$ for some $\xi \geq 1$. Let us write $u_n = u_n \restriction \alpha_0$ and $r_n = r_n \restriction \alpha_0$ for all n.

Claim. $\mathrm{crit}(u_n) \to \infty$ as $n \to \infty$.

Proof. See [3]. The only additional point here is that if $e = \mathrm{crit}(u_n)$ for infinitely many n, then $(e)_0 \neq 0$. □

It follows from the claim that for all e, $\psi_e(r_n^*)$ is eventually constant as $n \to \infty$. From this we get that $\lim^*_{n\to\infty} r_n = r$, for some $r \in A$ such that $p \subseteq r$. We shall show that r is a play by σ, so that σ was not winning for II is $\mathcal{G}(T)$, a contradiction.

Let β be least such that $r \restriction \beta + 1$ is not by σ. Fix η a limit ordinal such that $\beta = \eta + \omega$; such an η must exist because σ is for II, who only moves at stages of the form $\eta + \omega$. Let

$$\beta = |k|_{F(r)}$$

and

$$\alpha = \text{ eventual value of } |k|_{F(r_n)} \text{ as } n \to \infty.$$

Since the r_n^* converge in the scales $\vec{\theta}^5$, we have that $\alpha = \mu + \omega$ for some limit ordinal μ. We now look at how column α of our diagram was constructed. Since r_n^* converged in $\vec{\theta}^0$ and $\vec{\theta}^1$, we have

$$\lim^*_{n\to\infty} r_n \restriction \alpha = r \restriction \beta.$$

Since the r_n^* converge in $\vec{\theta}^6$, which is very good, the a_n's defined at round α are eventually constant, with value $a = r(\beta)_0$. But then at round α in the construction we set $d = \sigma((r \restriction \beta)^\frown\langle a\rangle)$. Moreover, Case 1 must have applied at α, as otherwise $\mathrm{crit}(u_n)$ would have a finite lim inf. Thus $r_n(\gamma) = \langle a, d\rangle$ for all sufficiently large n. Since $\vec{\theta}^6$ is very good, $r(\beta) = \langle a, d\rangle$, so that $r \restriction (\beta + 1)$ is by σ, a contradiction. This completes the proof of 2.4. $\square$

Let U be the tree of the semiscale $\vec{\phi}$ given by 2.4, and let $\vec{\theta}$ be the scale of U. One can easily check that, since $\Game^{\omega_1}$(open-analytical) is closed under real quantification, the θ_i are $\Game^{\omega_1}$(open-analytical) norms, uniformly in i. This completes the proof of 2.1. $\square$

There are some awkward features of our proof of 2.1. First, it only applies directly to games in which II is restricted to playing from $\{0, 1\}$. Second, our comparison games seem to only yield a semiscale directly, and not a scale. This is connected to the fact that we only show that if $p_n \to p$ modulo our semiscale, then II has no winning strategy in $\mathcal{G}(T)$ from p; we do not construct a winning strategy for I from p. One could probably obtain a more direct proof by bringing in the construction of definable winning strategies for I. In this connection, one has

Theorem 2.5 *Assume that ω_1-open-projective determinacy holds, and that* I *has a winning strategy in $\mathcal{G}(T)$, where T is an ω_1-tree which is definable over $(H_{\omega_1}, \in)$; then* I *has a $\Game^{\omega_1}$(open-analytical) winning strategy in $\mathcal{G}(T)$.*

One can prove 2.5 by modifying the proof of Theorem 2 of [3], in the same way that we modified the proof of Theorem 1 of [3] in order to prove 2.1.

References

[1] I. Neeman, *Games of length ω_1*, J. of Math. Logic, to appear.

[2] I. Neeman, *The determinacy of long games*, Walter de Gruyter, Berlin, 2004.

[3] J.R. Steel, *Long games*, in A.S. Kechris, D.A. Martin, and J.R. Steel (Eds.) Cabal Seminar 81-85, Lecture Notes in Mathematics 1333, Springer-Verlag, Berlin (1988), 56-97.