Notes on work of Jackson and Sargsyan

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March 2019

0 Introduction

We assume AD^+ throughout. Here is a nice conjecture on the relationship between Suslin cardinals and mouse-limits.

Conjecture. Let (P, Σ) be a mouse pair, and let κ be a cardinal of V such that $\kappa < o(M_\infty(P, \Sigma))$; then the following are equivalent:

(1) κ is a Suslin cardinal,

(2) κ is a cutpoint of $M_\infty(P, \Sigma)$.

The conjecture would imply that under HPC, the Suslin cardinals are precisely the cardinals of V that are cutpoints of the HOD-sequence. That (2) implies (1) follows easily from work in [8]:

Lemma 0.1. *Let (P, Σ) be a mouse pair, and $\kappa < o(M_\infty(P, \Sigma))$ be a cardinal of V. Suppose κ is a cutpoint of $M_\infty(P, \Sigma)$; then κ is a Suslin cardinal.*

We shall prove the lemma below.

Recent work of Jackson and Sargsyan gets us a lot closer to a proof of the converse direction. These notes are mostly an account of that work.

Definition 0.2. For any κ, meas_κ is the collection of all ultrafilters on κ

Of course, each $U \in \text{meas}_\kappa$ is countably complete, and Rudin-Kielser reducible to the Martin measure on degrees if $\kappa < \theta$.

G. Sargsyan recently proved the following.

Theorem 0.3 (Sargsyan [3])**.** *Assume AD^+, and let (P, Σ) be a mouse pair. Let E be an extender of the sequence of $M_\infty(P, \Sigma)$ with critical point κ, and such that*

(1) κ is a cutpoint of $M_\infty(P, \Sigma)$, and

(2) E is total on $M_\infty(P, \Sigma)$, and $\kappa < \rho_n(M_\infty(P, \Sigma))$, for $n = k(P)$.

Then there is a $U \in meas_\kappa$ such that if $j_U\colon V \to \mathrm{Ult}(V,U)$ and $i_E\colon M_\infty(P,\Sigma) \to \mathrm{Ult}(M_\infty(P,\Sigma),E)$ are the canonical embeddings, then

$$j_U \restriction M_\infty(P,\Sigma) = \sigma \circ i_E,$$

for some elementary $\sigma\colon \mathrm{Ult}(M_\infty(P,\Sigma),E) \to j_U(M_\infty(P,\Sigma))$, and hence

$$\lambda_E \leq j_U(\kappa).$$

See [3]. (Sargsyan did not state his result in this generality, but this is what his proof gives.) Using the known connections between Suslin cardinals, measures, and Martin classes (see [1][§3]), Theorem 0.3 yields at once the author's theorem that under $\mathsf{AD}_{\mathbb{R}} + \mathsf{HPC}$, every point θ_α in the Solovay sequence is a cutpoint of the HOD-sequence. [1] (See [8].) The resulting proof is simpler and more general than that of [8].

Jackson has recently observed that the results on Martin classes of [1][§3] can be extended so as to prove the following.

Theorem 0.4 (Jackson)**.** *Assume AD^+. Let κ be a limit of Suslin cardinals of uncountable cofinality, and λ the least Suslin cardinal $> \kappa$; then for any ultrafilter U on κ, $j_U(\kappa) < \lambda$.*

We remark that if λ is Suslin, then $\lambda \leq \sup(\{j_U(\kappa) \mid \kappa < \lambda \wedge U \in \mathrm{meas}_\kappa\})$. This comes from Martin-Solovay construction of a scale on $\neg p[T]$, where T is weakly homogeneous. We do not know whether the reverse inequality holds at all Suslin cardinals λ. If λ is a limit of Suslins, it is trivial. If λ is the next Suslin after a limit of Suslins, then the theorem above says a lot, but does not fully answer the question.

Definition 0.5. For any premouse Q and $\kappa < o(Q)$,

(a) $o(\kappa)^Q$ is the strict sup of all η such that $\mathrm{crit}(E^Q_\eta) = \kappa$. If there are no such η, then $o(\kappa)^Q = 0$.

(b) κ is *Q-regular* iff there is no $\eta < \kappa$ and total $\Sigma_{k(Q)}(Q)$ function $f\colon \eta \to \kappa$ with range cofinal in κ.

(c) κ is *Q-measurable* iff $\kappa < \rho_{k(Q)}(Q)$, and $o(\kappa)^Q\eta \geq (\kappa^+)^Q$.

Coherence implies that if κ is a cutpoint of Q, so is $o(\kappa)^Q$. Our notion of regularity involves all functions that might be used in some nondropping ultrapower of Q. Thus we have by the usual "regulars are measurable" argument

Lemma 0.6. *Let (P,Σ) be a mouse pair, $M_\infty = M_\infty(P,\Sigma)$, and $\kappa < o(M_\infty)$ have uncountable cofinality in V; then κ is M_∞-regular iff κ is M_∞-measurable.*

[1] It is important here that we are talking about cutpoints with respect to extenders on the HOD-sequence. We do not have a proof that every θ_α is a cutpoint with respect to extenders belonging to HOD. The results of [4] would seem to be relevant there.

Putting the two theorems above together, with some sauce from [8], we get the following.

Theorem 0.7. *Assume* AD^+*, let* (P,Σ) *be a mouse pair, and let* $M_\infty = M_\infty(P,\Sigma)$*. Let* $\kappa < o(M_\infty)$ *be a limit of Suslin cardinals such that* $\mathrm{cof}(\kappa) > \omega$ *in* V*; then*

(1) κ *is a limit of cutpoints of* M_∞*, and*

Suppose $\kappa^+ \leq o(M_\infty)$*, and let* λ *be the least Suslin cardinal* $> \kappa$*; then*

(2) there is a cutpoint μ *of* M_∞ *such that* $\kappa \leq \mu < (\kappa^+)^V$ *and* $o(\mu)^{M_\infty} = \lambda$*,*

(3) if $o(\kappa)^{M_\infty} \geq (\kappa^+)^V$*, then* $o(\kappa)^{M_\infty} = \lambda$*, and*

(4) if S_κ *is closed under* $\forall^{\mathbb{R}}$*, then* $o(\kappa)^{M_\infty} = \lambda$*.*

Some of these results were proved in [8] in the case that (P,Σ) is a pointclass generator. From this we get at once

Corollary 0.8. *Assume* $\mathsf{AD}_{\mathbb{R}} + \mathsf{HPC}$*, and let* κ *be a limit of Suslin cardinals of uncountable cofinality, and regular in HOD, and let* λ *be the least Suslin cardinal* $> \kappa$*; then*

(1) $(\kappa^+)^{HOD} \leq o(\kappa)^{HOD} \leq \lambda$*,*

(2) there is a cutpoint μ *of HOD such that* $\kappa \leq \mu < (\kappa^+)^V$ *and* $o(\mu)^{HOD} = \lambda$*,*

(3) if $o(\kappa)^{HOD} \geq (\kappa^+)^V$*, then* $o(\kappa)^{HOD} = \lambda$*, and*

(4) if S_κ *is closed under* $\forall^{\mathbb{R}}$*, then* $o(\kappa)^{HOD} = \lambda$*.*

If κ is a countable cofinality limit of Suslins, then κ^+ is the next Suslin (and somewhat like ω_1). See [1][3.28]. In this case we have

Theorem 0.9. *Let* (P,Σ) *be a mouse pair,* $M_\infty = M_\infty(P,\Sigma)$*, and* κ *be a limit of Suslin cardinals of countable* V*-cofinality. Suppose* $(\kappa^+)^V \leq o(M_\infty)$*; then*

(1) κ *and* $(\kappa^+)^V$ *are limits of cutpoints in* M_∞*, and*

(2) $(\kappa^+)^V < \rho_{k(P)}(M_\infty)$*, and* $(\kappa^+)^V$ *is the critical point of a total extender from the* M_∞*-sequence.*

And then of course there is a corollary for HOD parallel to 0.8.

These results seem close to a proof of the conjecture. What's missing is the analog of Jackson's result on measure-bounding for the Suslin cardinals corresponding to higher levels of a projective-like hierarchy. For the ordinary projective hierarchy, Jackson has proved these as part of his computation of the projective ordinals. But perhaps the full force of this machinery is not needed. We do have

Theorem 0.10. *The conjecture holds when* κ *is one of the* $\boldsymbol{\delta^1_{2n+1}}$*'s or their cardinal predecessors.*

Theorem 0.10 was known for various natural (P, Σ) by other means already. Theorem 0.3, [8], and Jackson's results on measure bounding in the projective hierarchy yield a different, more general proof.[2]

In this note we shall prove the results above. We emphasize, however, that most of what is new here is due to Jackson and Sargsyan. We wrote this note in Fall 2018, inspired by an earlier version of Sargsyan's [3].

1 Proof of Theorem 0.3

Let (P, Σ) be a mouse pair, $\mathcal{F}(P, \Sigma)$ the directed system of all its nondropping iterates, and $M_\infty = M_\infty(P, \Sigma)$ the direct limit of $\mathcal{F}(P, \Sigma)$. For the associated iteration maps of the system we write $\pi_{Q,R} \colon Q \to \mathbb{R}$ and $\pi_{Q,\infty} \colon Q \to M_\infty$. It's ok here to drop mention of the strategy of Q, since we are dealing exclusively with tails of a single positional strategy Σ. Let $k = k(P)$. [3] Let E be an extender on the sequence of M_∞, and $\kappa = \mathrm{crit}(E)$. Suppose that κ is a cutpoint of P (and hence a limit of cutpoints), and that E is total on P and $\kappa < \rho_k(P)$. We want to embed $\mathrm{Ult}(M_\infty, E)$ into $j_U(M_\infty)$, for some ultrafilter U on κ.

By replacing (P, Σ) with an iterate of itself, we may assume $E \in \mathrm{ran}(\pi_{P,\infty})$. For any $(R, \Sigma_R) \in \mathrm{cof}(P, \Sigma)$, let

$$\pi_{R,\infty}(E_R) = E,$$

and

$$\pi_{R,\infty}(\kappa_R) = \kappa.$$

If d is a Turing degree and $Q \in \mathrm{HC}$, we write $Q \leq d$ to mean that Q is coded by a real recursive in d. (Fix some natural coding system.) Let

$$\mathcal{F}(d) = \{(Q, \Sigma_Q) \mid Q \leq d \wedge (Q, \Sigma_Q) \in \mathcal{F}(P, \Sigma)\},$$

and

$$M_d = \text{ result of simulaneously comparing all } (Q, \Sigma_Q) \in \mathcal{F}(d).$$

We note here that by [5], Σ is positional. It follows that comparisons between iterates of (P, Σ) never encounter strategy disagreements, and so can be done by iterating away least extender disagreements as usual. The simultaneous comparsion referred to above proceeds by iterating away least extender disagreements. It does not depend on any enumeration of d, just d itself. Set also

$$\Sigma_d = \Sigma_{M_d},$$

so that $(M_d, \Omega_d) \in \mathcal{F}(P, \Sigma)$. Let $(E_d, \kappa_d) = (E_{M_d}, \kappa_{M_d})$, and

$$Q_d = M_d || \mathrm{lh}(E_d),$$

[2]So for example, when P is M_1 cut at its Woodin, and Σ is its canonical strategy, we get a new proof that $o(M_\infty(P, \Sigma)) = \aleph_\omega$, with the least strong of M_∞ being $< \omega_2$. And in fact, something similar must happen for any (P, Σ) such that $o(M_\infty(P, \Sigma)) \geq \omega_2$.

[3]This is the quantifier level at which we are condisdering P. P is always $k(P)$ sound, but it may not be $k(P) + 1$ sound. See [7][§1].

and

$$\Omega_d = (\Sigma_d)_{Q_d}$$

be the strategy for Q_d that is part of Σ_d. Our "$||$" notation indicates that Q_d is passive, that is, the last extender predicate E_d has been removed.

Claim 1. $M_\infty(Q_d, \Omega_d) \trianglelefteq M_\infty$.

Proof. Let $R = Ult(M_d, E_d)$ and $S = \mathrm{Ult}(R, F)$, where F is the order zero total measure on $\lambda(E_d) = i^{M_d}_{E_d}(\kappa_d)$. Then $(S, \Sigma_S) \in \mathcal{F}(P, \Sigma)$, and (Q_d, Ω_d) is a cardinal cutpoint initial segment of (S, Σ_S). The claim follows. □

We can now define our ultrafilter U on κ. Let $\bar{\pi}_{d,\infty} = \pi^{\Omega_d}_{Q_d,\infty}$. For $A \subseteq \kappa$,

$$A \in U \text{ iff } \forall^* d(\bar{\pi}_{d,\infty}(\kappa_d) \in A).$$

Here $\forall^* d$ refers to the Martin measure. U is clearly a countable complete ultrafilter on κ. We must now define the desired $\sigma\colon \mathrm{Ult}(M_\infty, E) \to j_U(M_\infty)$. Of course, the definition will be in the form $\sigma(i^{M_\infty}_E(f)(a)) = j_U(f)(\sigma(a))$. We just have to figure out what $\sigma(a)$ is.

Fix an $a \in [\lambda(E)]^{<\omega}$, and let (R, Σ_R) be such that $a \in \mathrm{ran}(\pi_{R,\infty})$. Say

$$\pi_{R,\infty}(a_R) = a.$$

We shall define a function f^a_R, and show that $[f^a_R]_U$ is independent of the R we have chosen. We then set $\sigma(a) = [f^a_R]_U$. Towards defining f^a_R, let d be any degree such that $R \leq d$, and set

$$a_d = \pi_{R,M_d}(a_R) = \pi^{-1}_{M_d,\infty}(a).$$

The main claim is the following.

Claim 2. Let $a \in [\lambda(E)]^{<\omega}$, and suppose $R \leq c$ and $R \leq d$. Suppose $\bar{\pi}_{c,\infty}(\kappa_c) = \bar{\pi}_{d,\infty}(\kappa_d)$; then $\bar{\pi}_{c,\infty}(a_c) = \bar{\pi}_{d,\infty}(a_d)$.

Proof. Let $\mathcal{T}$ be the normal tree by Σ_R from R to M_c, and let α be least such that $\mathrm{lh}(E^{\mathcal{T}}_\alpha) \geq \mathrm{lh}(E_c)$. Since E_c is on the last model of $\mathcal{T}$, $\mathrm{lh}(E^{\mathcal{T}}_\alpha) > \mathrm{lh}(E_c)$. Also, $E_c \in \mathrm{ran}(i^{\mathcal{T}}_{0,\infty})$, so α is on the main branch of $\mathcal{T}$, and $\mathrm{crit}(i^{\mathcal{T}}_{\alpha,\infty}) > \mathrm{lh}(E_c)$. Note that $Q_c \trianglelefteq \mathcal{M}^{\mathcal{T}}_\alpha$.

Similarly, let $\mathcal{U}$ be the normal tree by Σ_R from R to M_d, and let β be least such that $\mathrm{lh}(E^{\mathcal{U}}_\beta) \geq \mathrm{lh}(E_d)$. Since E_d is on the last model of $\mathcal{U}$, $\mathrm{lh}(E^{\mathcal{U}}_\beta) > \mathrm{lh}(E_d)$. Again, β is on the main branch of $\mathcal{U}$, $\mathrm{crit}(i^{\mathcal{U}}_{\beta,\infty}) > \mathrm{lh}(E_d)$, and $Q_d \trianglelefteq \mathcal{M}^{\mathcal{U}}_\beta$.

Now notice that $M_\infty(Q_c, \Omega_c) = M_\infty(Q_d, \Omega_d)$. This is because both are cutpoint initial segments of $M_\infty(P, \Sigma)$, and both have a top block that begins at the same place, namely $\bar{\pi}_{c,\infty}(\kappa_c) = \bar{\pi}_{d,\infty}(\kappa_d)$. It follows that (Q_c, Ω_c) is mouse equivalent to (Q_d, Ω_d) (see [8][2.2]). They compare by iterating away least extender disagreements, because we are working with tails of a single positional strategy. Let $\mathcal{T}_1$ on Q_c and $\mathcal{U}_1$ on Q_d be the normal trees with common last model S that we get from this comparison. It is enough to see that

$$i^{\mathcal{T}_1}(a_c) = i^{\mathcal{U}_1}(a_c),$$

where these are the main branch embeddings of $\mathcal{T}_1$ and $\mathcal{U}_1$. (Note $i^{\mathcal{T}_1} = \pi^{\Omega_c}_{Q_c,S}$ and $i^{\mathcal{U}_1} = \pi^{\Omega_d}_{Q_d,S}$.)

To see this, consider the normal trees

$$\mathcal{T}_0 = \mathcal{T} \restriction (\alpha+1)^\frown\langle E_c\rangle$$

and

$$\mathcal{U}_0 = \mathcal{U} \restriction (\beta+1)^\frown\langle E_d\rangle.$$

It is important here that we are talking about normal extensions; E_c may not be applied to $\mathcal{M}^{\mathcal{T}}_\alpha$, but instead some earlier model. Letting $N_0 = \mathcal{M}^{\mathcal{T}_0}_{\alpha+1}$ and $N_1 = \mathcal{M}^{\mathcal{U}_0}_{\beta+1}$ be the last models, we have that Q_c is a cardinal cutpoint initial segment of N_0, and $o(Q_c) < \rho_k(N_0)$, and similarly for Q_d and N_1. Thus $\mathcal{T}_1$ and $\mathcal{U}_1$ can be considered as normal, nondropping[4] trees on N_0 and N_1. Let us do that. Let

$$X = X(\mathcal{T}_0, \mathcal{T}_1),$$

and

$$Y = X(\mathcal{U}_0, \mathcal{U}_1)$$

be the full normalizations of the two stacks, so that X and Y are normal trees on R by Σ_R. (See [9] or [5].) We can write

$$X = X_0{}^\frown\langle F\rangle$$

and

$$Y = Y_0{}^\frown\langle G\rangle,$$

where X_0 and Y_0 have last models N_0^* and N_1^* respectively, both extend S, and F and G are the extenders with index $o(S)$ in the two models. Now note that the generators for the branch extender R-to-N_0^* in X_0 are contained in $o(S)$, as are the generators of R-to-N_1^* in Y_0. So both are trees by Σ_R using only extenders of length $< o(S)$, so in fact,

$$X_0 = Y_0$$

and $N_0^* = N_1^*$, and $i^{X_0} = i^{Y_0}$.

Let Φ and Ψ be the weak tree embeddings of $\mathcal{T}_0$ and $\mathcal{U}_0$ into X and Y that come from full normalization. We have

$$t^\Phi_{\alpha+1} \colon N_0 \to N_0^*$$

and

$$t^\Psi_{\beta+1} \colon N_1 \to N_1^*,$$

from that process, with

$$t^\Phi_{\alpha+1} \restriction \mathrm{lh}(E_c) = t^\Phi_\alpha \restriction \mathrm{lh}(E_c) = i^{\mathcal{T}_1} \restriction \mathrm{lh}(E_c),$$

and

$$t^\Psi_{\beta+1} \restriction \mathrm{lh}(E_d) = t^\Psi_\beta = i^{\mathcal{U}_1} \restriction \mathrm{lh}(E_d).$$

[4]The reader must have figured out by now that this means the main branch does not drop.

Also,

$$i^{X_0} = t^{\Phi}_{\alpha} \circ i^{\mathcal{T}_0}_{0,\alpha}$$

and

$$i^{Y_0} = t^{\Psi}_{\beta} \circ i^{\mathcal{U}_0}_{0,\beta},$$

by the way normalization works. Letting a_R be the preimage of a in R, we then have

$$\begin{aligned} i^{\mathcal{T}_1}(a_c) &= t^{\Phi}_{\alpha}(a_c) \\ &= t^{\Phi}_{\alpha} \circ i^{\mathcal{T}_0}_{0,\alpha}(a_R) \\ &= i^{X_0}(R) = i^{Y_0}(a_R) \\ &= t^{\Psi}_{\beta} \circ i^{\mathcal{U}_0}_{0,\beta}(a_R) \\ &= t^{\Psi}_{\beta}(a_d) \\ &= i^{\mathcal{U}_1}(a_d), \end{aligned}$$

as desired. This proves the claim. □

Remark. See [8][Lemma 2.24] for an argument that overlaps with this one.

Let us define, for any $\beta < \kappa$,

$$f^a_R(\beta) = b \text{ iff } \exists d(R \leq d \wedge \beta = \bar{\pi}_{0,\infty}(\kappa_d) \wedge b = \bar{\pi}_{0,\infty}(a_d)).$$

Claim 2 implies that f^a_R is a function. It is clear that $\mathrm{dom}(f^a_R) \in U$.

Claim 3. Let R and S be such that $a \in \mathrm{ran}(\pi_{R,\infty})$ and $a \in \mathrm{ran}(\pi_{S,\infty})$; then $[f^a_R]_U = [f^a_S]_U$.

Proof. For U a.e. β, there is a d such that $R \leq d$, $S \leq d$, and $\beta = \kappa_d$. For any such β and d, $f^a_R(\beta) = \bar{\pi}_{d,\infty}(a_d) = f^a_S(\beta)$. □

We shall set $\sigma(a) = [f^a_R]_U$. This leads to $\sigma([a,g]^{M_\infty}_E) = j_U(g)([f^a_R]_U)$, or in other words,

$$\sigma([a,g]^{M_infty}_E) = [g \circ f^a_R]_U.$$

The following claim implies that this works.

Claim 4. Let $\mathrm{Ult}(M_\infty, E) \models \varphi[[a_0, g_0], ..., [a_n, g_n]]$, and let $a_i \in \mathrm{ran}(\pi_{R_i,\infty})$ for all $i \leq n$. Then for U-a.e. β, $M_\infty \models \varphi[g_0(f^{a_0}_{R_0}(\beta)), ..., g_n(f^{a_n}_{R_n}(\beta))]$.

Proof. Let us assume $n = 0$, and write $g = g_0$, $a = a_0$, and $R = R_0$. By Claim 3, we may assume that $g \in \mathrm{ran}(\pi_{R,\infty})$. It is enough to show that whenever $R \leq d$, then $M_\infty \models \varphi[g(\bar{\pi}_{d,\infty}(a_d)]$.

Let $g = \pi_{R,\infty}(g_R)$, $a = \pi_{R,\infty}(a_R)$, and $E = \pi_{R,\infty}(E_R)$. We have that for $(E_R)_{a_R}$ a.e. U, $R \models \varphi[g_R(u)]$. Now let $R \leq d$, and set $g_d = \pi_{R,M_d}(g_R)$. Again, we have that for $(E_d)_{a_d}$ a.e. u, $M_d \models \varphi[g_d(u)]$. It follows that

$$\mathrm{Ult}(M_d, E_d) \models \varphi[i^{M_d}_{E_d}(g)(a_d)].$$

Letting $S = \mathrm{Ult}(M_d, E_d)$, we have $\pi_{M_d,\infty} = \pi_{S,\infty} \circ i^{M_d}_{E_d}$, so

$$M_\infty \models \varphi[g(\pi_{S,\infty}(a_d)].$$

But $Q_d = S|lh(E_d)$, and $\lambda(E_d)$ is a cutpoint of S, so by strategy coherence

$$\pi_{S,\infty} \restriction \lambda(E_d) = \bar{\pi}_{d,\infty} \restriction \lambda(E_d).$$

Thus $M_\infty \models \varphi[g(\bar{\pi}_{d,\infty}(a_d)]$, as desired. $\square$

By Claim 4, the map $\sigma([a,g]^{M_\infty}_E) = [g \circ f^a_R]_U$ is well defined and elementary. Written otherwise, $\sigma(i^{M_\infty}_E(g)(a)) = j_U(g)([f^a_R])$. Applied to constant functions g, this tells us $j_U \restriction M_\infty = \sigma \circ i^{M_\infty}_E$. Evaluating at κ, we see that $i^{M_\infty}_E(\kappa) \leq j_U(\kappa)$.

2 Proof of Theorem 0.4

Definition 2.1. S_κ is the pointclass of κ-Suslin sets.

Let κ be a limit of Suslin cardinals, and $\mathrm{cof}(\kappa) > \omega$. Put

$$\Delta = \bigcup_{\alpha<\kappa} S_\alpha.$$

$\kappa = \delta(\Delta)$ is the sup of the lengths of prewellorderings in Δ, as well as its Wadge rank. (See the proof of 3.8 of [1].) Let Γ be the boldface pointclass such that

$$\Delta = \Gamma \cap \breve{\Gamma} \text{ and } \breve{\Gamma} \text{ has the Separation property.}$$

The paper [6] shows there is such a Γ, identifies Γ as the class of $\mathbf{\Sigma^1_1}$-bounded unions of sets in $\bigcup_{\alpha<\kappa} S_\alpha$, and shows $\forall^{\mathbb{R}}\Gamma \subseteq \Gamma$. Jackson has shown that Γ is precisely the class of all $p[T]$, for T a homogeneous tree on $\omega \times \kappa$, and that it has the scale property. See [1][3.8]. It is also shown there that $S_\kappa = \exists^{\mathbb{R}}\Gamma$. Another somewhat useful fact is that there is a regular Γ norm φ on a complete Γ set such that $\leq_\varphi$ has order type κ. (See [1][2.22].)

Let us fix such a Γ norm $\varphi \colon B \to \mathbb{R}$. Using φ and the uniform coding lemma (see [2]), we get a coding of subsets of κ. To be precise, let $B_\alpha = \{(x,y) \mid \varphi(x) \leq \varphi(y) \leq \alpha\}$. For any $A \subset \kappa$, there is a real x and Σ^1_1 formula ψ such that for all $\alpha < \kappa$ and y such that $\varphi(y) = \alpha$ and $\gamma > \alpha$,

$$\alpha \in A \Leftrightarrow \psi(y, B_\gamma, x).$$

(B can occur negatively in ψ.) We can assume ψ is fixed for all x by using a universal formula. For any real x, let

$$\alpha \in A_x \text{ iff } \exists\gamma > \alpha \exists y(\varphi(y) = \alpha \wedge \psi(y, B_\gamma, x)).$$

So $P(\kappa) = \{A_x \mid x \in \mathbb{R}\}$. Using the Godel pairing we let

$$f_x = \{(\alpha,\beta) \mid (\alpha,\beta) \in A_x\}.$$

Of course, f_x may not be a function. We say x is *single valued* if f_x is a function. (It need not be total, however.)

We define the Martin class, or envelope, of Γ by

$$A \in \Lambda(\Gamma, \kappa) \text{ iff } \exists \langle A_\alpha \mid \alpha < \kappa \rangle [\forall \alpha < \kappa (A_\alpha \in \Delta) \text{ and } \\ \forall^* d \exists \alpha < \kappa (A \cap \{x \mid x \leq d\} = A_\alpha \cap \{x \mid x \leq d\}).$$

The main thing is

Claim 1. Let $U \in \text{meas}_\kappa$, and put $x \prec y$ iff (x and y are single valued and defined U-a.e., and $[f_x]_U \leq [f_y]_U$); then $\prec$ is in $\Lambda(\Gamma, \kappa)$.

Proof. For $\gamma < \beta < \kappa$, put

$$(x, y) \in A_{\beta,\gamma} \Leftrightarrow (\gamma \in \text{dom}(f_x) \cap \text{dom}(f_y) \wedge f_x(\gamma) \leq f_y(\gamma) < \beta \\ \wedge f_x \cap \beta \times \beta \text{ and } f_y \cap \beta \times \beta \text{ are single valued.}$$

It is enough to show that for any d, there is a (β, γ) such that $\prec$ agrees with $A_{\beta,\gamma}$ on the reals $\leq d$. But by countable completeness, we can find γ such that for all single valued $x \leq d$, $\gamma \in \text{dom}(f_x)$ iff $\text{dom}(f_x) \in U$. Similarly, we can arrange that for $x, y \leq d$ single-valued with domains in U, $f_x(\gamma) \leq f_y(\gamma)$ iff $[f_x]_U \leq [f_y]_U$. Finally, since κ has uncountable cofinality, we can choose β large enough that all relevant $f_x(\gamma)$ are below β, and any non-single-valued $x \leq \beta$ are such that $f_x \cap \beta \times \beta$ is not single valued. This proves the Claim. $\square$

Let λ be the least Suslin cardinal $> \kappa$. As shown in [1], the universal $\check{S}_\kappa$ set has a semi-scale all of whose norm relations are in the envelope $\Lambda(\Gamma, \kappa)$.[5]

If S_κ is closed under $\forall^\mathbb{R}$, or equivalently $S_\kappa = \Gamma$, we get $\Lambda(\Gamma, \kappa)$ is closed under real quantifiers. Martin's non-uniformizability result then shows that λ is at least prewellordering ordinal of $\Lambda(\Gamma, \kappa)$. (See [1][3.17].) Combined with Claim 1, this gives $\lambda \geq \sup(\{j_U(\kappa) \mid U \in \text{meas}_\kappa\})$. So λ, this sup, and the prewellordering ordinal of Λ coincide.

Now let us assume that Γ is not closed under $\exists^\mathbb{R}$, and look at the projective-like hierarchy above it. We write $\Pi_1 = \Gamma$, $\Sigma_1 = \check{\Gamma}$, and $\Pi_{n+1} = \forall^\mathbb{R}\Sigma_n$ and $\Sigma_{n+1} = \exists^\mathbb{R}\Pi_n$. For $n > 1$, these pointclasses have the usual closure properties of the levels of the projective hierarchy. By periodicity, the Π_{2n+1} and Σ_{2n+2} have the scale property. Since we are asuming $S_\kappa \neq \Pi_1$, we get $S_\kappa = \Sigma_2$. It follows by the ordinary projective hierarchy arguments that λ has cofinality ω, $S_\lambda = \Sigma_3$, and λ^+ is the next Suslin after λ, and the prewellordering ordinal of Π_3.

Remark. In the present case, $S_\kappa = \Sigma_2$ is the class of all κ-length unions of sets in Δ. It is therefore properly included in $\Lambda(\Gamma, \kappa)$. What Martin's proof shows is that there is a Π_2 relation with no uniformization in $\Lambda(\Gamma, \kappa)$.

But now let $\prec$ be any prewellorder in $\Lambda(\Gamma, \kappa)$. If $\lambda^+ \leq |\prec|$, then $\prec$ is not λ-Suslin by Kunen-Martin. It follows by Wadge that a universal Π_3 set is Wadge below $\prec$. But we can uniformize every Π_2 relation in Π_3, since the latter has the scale property. This contradicts

[5] Jackson and Woodin showed there is a self-justifying system sealing the envelope, in fact.

Martin's theorem. We have proved that $|j_U(\kappa)| \leq \lambda$ for all $U \in \text{meas}_\kappa$. (But the sup might be λ^+.)

Now let us assume κ is regular. This makes Γ a nicer pointclass, closed under countable unions and intersections, for example. Also, κ has the strong partition property from the usual arguments using the uniform coding lemma. A general fact is that if κ is any cardinal with the strong partition if U is semi-normal, that is gives every club set measure one, then $j_U(\kappa)$ is regular. So, if $j_U(\kappa) \geq \lambda$, then $j_U(\kappa) \geq \lambda^+$. We have just shown this is not the case.

Finally, suppose that κ is singular. Let W be the ω-club ultrafilter on $\text{cof}(\kappa)$, which exists because we are in the range of the HOD-analysis. Jackson shows that $j_{U\times W}(\kappa)$ is a cardinal. (Proof to come!) This shows $j_{U\times W}(\kappa) < \lambda$, so $j_U(\kappa) < \lambda$.

This completes the proof of Theorem 0.4.

3 Proofs of 0.1, 0.7, and 0.9

Proof of 0.1.

We are given a mouse pair (P, Σ), and $\kappa < o(M_\infty(P, \Sigma)$ such that κ is a cardinal of V, and a cutpoint of $M_\infty(P, \Sigma)$. By [8][2.19], $|\tau_\infty(P, \Sigma)|$ is a Suslin cardinal, and $|\tau_\infty(P, \Sigma)| = |o(M_\infty(P, \Sigma)| \geq \kappa$. So if $\tau_\infty(P, \Sigma) < (\kappa^+)^V$, then κ is a Suslin cardinal, as desired. So assume $(\kappa^+)^V \leq \tau_\infty(P, \Sigma)$.

Set $M_\infty = M_\infty(P, \Sigma)$. Suppose that $o(\kappa)^{M_\infty} \geq \tau_\infty(P, \Sigma)$. Since κ is a cutpoint of M_∞, this implies that (P, Σ) has a top block, and $\beta_\infty(P, \Sigma) = \kappa$. Then by [8][2.27], κ is a Suslin cardinal, as desired.

So suppose that $o(\kappa)^{M_\infty} < \tau_\infty(P, \Sigma)$. The following little lemma is useful.

Lemma 3.1. *Let (R, Ω) be a mouse pair, $R_\infty = M_\infty(R, \Omega)$, and $k = k(R)$. Let*

$$\gamma = \sup\{\eta, o(\eta)^{R_\infty}\},$$

where η is a cardinal of R_∞, and

$$\xi = (\gamma^+)^{R_\infty}.$$

Suppose $\xi \leq \rho_k(R_\infty)$; then there is a (Q, Ψ) such that

(a) $M_\infty(Q, \Psi) = R_\infty|\xi$, and

(b) $\tau_\infty(Q, \Psi) \leq \gamma$.

Thus $|\gamma|$ is a Suslin cardinal.

Proof. By replacing (R, Ω) with an iterate of itself, we may assume that we have $\bar{\eta}, \bar{\gamma}$, and $\bar{\xi}$ such that $\pi_{R,\infty}(\langle\bar{\eta}, \bar{\gamma}, \bar{\xi}\rangle) = \langle\eta, \gamma, \xi\rangle$. By coherence, $\bar{\xi}$ is a cutpoint of R. Also, $\bar{\xi} \leq \rho_k(R)$. $\bar{\xi}$ is regular in R, so if it were not R-regular, we would have $\bar{\xi} = \rho_k(R)$ and some $\Sigma_k(R)$ partial f with $\text{dom}(f) \subseteq \bar{\gamma}$ and $\text{ran}(f)$ cofinal in $\bar{\xi}$. This easily yields $\rho_k(R) \leq \bar{\gamma}$, contradiction. Thus $\bar{\xi}$ is R-regular.

But then we can take $Q = R|\bar{\xi}$ and $\Psi = \Omega_Q$. $\square$

Now let $\gamma = o(\kappa)^{M_\infty}$, and $\xi = (\gamma^+)^{M_\infty}$. We are assuming $\gamma < \tau_\infty(P,\Sigma)$, and $\tau_\infty(P,\Sigma) < \rho_k(M_\infty)$ by its definition, so $\xi \le \rho_k(M_\infty)$. Thus by the lemma, $|\gamma|$ is a Suslin cardinal, so we may assume $(\kappa^+)^V \le \gamma$, otherwise we're done. Let then (Q,Ψ) be such that $M_\infty(Q,\Psi) = M_\infty|\xi$. It is easy to see that $\kappa = \beta_\infty(Q,\Psi)$. Thus by [8][2.27], κ is a Suslin cardinal. This completes the proof of 0.1. $\square$

We turn to 0.7 and 0.9.

Let (P,Σ) be a mouse pair, and κ a limit of Suslin cardinals, and $\kappa < o(M_\infty)$ where $M_\infty = M_\infty(P,\Sigma)$. Replacing (P,Σ) with an iterate of itself, we may assume $\kappa = \pi_{P,\infty}(\kappa_P$. Let $\pi_{P,\infty}(\kappa_P) = \kappa$.

That κ is a limit of cutpoints in P was proved in [8]. (Let μ be least such that $\mu < \kappa$ and $o(\mu)^{M_\infty} \ge \kappa$. By Cor. 2.42 of [8], there are no Suslin cardinals strictly between μ and $o(\mu)^{M_\infty}$. Contradiction.) We can also prove it using the measure existence result 0.3, and a softer coarser form of the measure bounding result 0.4.

Proof of 0.7.

We assume $\mathrm{cof}(\kappa) > \omega$ and $\kappa^+ \le o(M_\infty)$. Thus $\kappa^+ \le \tau_\infty(P,\Sigma)$. Let λ be the least Suslin cardinal $> \kappa$, so that by [1][§3], $\mathrm{cof}(\lambda) = \omega$, and hence $\kappa^+ < \lambda$. Since $|\tau_\infty(P,\Sigma)|$ is a Suslin cardinal, $\lambda \le \tau_\infty(P,\Sigma)$. Thus $\lambda \le \rho_k(M_\infty)$, where $k = k(P)$.

By Lemma 3.1, there is no cutpoint ξ of M_∞ such that $\xi = (\gamma^+)^{M_\infty}$ for some γ, and $\kappa^+ < \xi < \lambda$. For otherwise, there would be Suslin cardinals in the interval (κ,λ). It follows that there is a $\mu < \kappa^+$ such that $o(\mu)^{M_\infty} \ge \lambda$. Since κ is a cutpoint, $\kappa \le \mu$.

By coherence, we get that if $(\kappa^+)^V \le o(\kappa)^{M_\infty}$, then $\lambda \le o(\kappa)^{M_\infty}$.

Let $\mu < \kappa^+$ be such that $o(\mu)^{M_\infty} \ge \lambda$. Let E be a total M_∞ extender with critical point μ. By Theorem 0.3, there is an ultrafilter U on μ such that $\lambda(E) \le j_U(\mu)$. But $|\mu| = \kappa$, so by Theorem 0.4, $j_U(\kappa) < \lambda$, and thus $j_U(\mu) < \lambda$. So $o(\mu)^{M_\infty} \le \lambda$, hence $o(\mu)^{M_\infty} = \lambda$.

Finally, suppose S_κ is closed under $\forall^{\mathbb{R}}$. We must see $o(\kappa)^{M_\infty} \ge \kappa^+$. If not, by Lemma 3.1 we get (Q,Ψ) such that $\kappa < o(M_\infty(Q,\Psi)) < \kappa^+$. Thus $\mathrm{Code}(\Psi)$ is κ-Suslin. But Ψ is a complete strategy, so

$$\mathcal{T} \text{ is not by } \Psi \text{ iff } \exists\alpha < \mathrm{lh}(\mathcal{T})(\Psi(\mathcal{T}\restriction\alpha) \ne [0,\alpha)_T).$$

Thus $\neg\,\mathrm{Code}(\Psi)$ is also κ-Suslin. Since S_κ is inductive like, we get that $\mathrm{Code}(\Psi) \in S_\alpha$ for some $\alpha < \kappa$, contrary to Kunen-Martin and the fact that $\kappa \le o(M_\infty(Q,\Psi)$. $\square$

Proof of 0.9

$\kappa < \rho_k(M_\infty)$ because we demanded $(\kappa^+)^V \le o(M_\infty)$. So if κ were measurable in M_∞, the set of images of iteration points of the order zero measure would have uncountable cofinality, contrary to $\mathrm{cof}(\kappa) = \omega$. $\kappa^+ = (\kappa^+)^V$ is regular, in fact measurable, in V because it is the prewellordering ordinal of a Π^1_1-like pointclass. So $\kappa^+ = o(M_\infty)$ is impossible, as $\pi_{P,\infty}``o(P)$ would be cofinal in κ^+. A similar argument shows κ^+ is measurable in M_∞.

Finally, suppose toward contradiction that $\kappa < \mu < \kappa^+$, and μ is a cutpoint of M_∞ such that $o(\mu)^{M_\infty} \ge \kappa^+$. Then $\mathrm{cof}(\mu) > \omega$ because μ is measurable in M_∞ by a total measure, and

$o(\mu)^{M_\infty} > \kappa^+$ by coherence and the fact that κ^+ is measurable in M_∞.. Applying the result of Sargsyan, out in V there is an ultrafilter U on μ such that $j_U(\mu) > \kappa^+$. But $|\mu| = \kappa$, so we can use a bijection to replace U with an ultrafilter W on κ such that $j_W = j_U$. Since $\mathrm{cof}(\kappa) = \omega$, we may assme that W is actually an ultrafilter on some $\eta < \kappa$.

κ is a limit of Suslin cardinals, so easy measure bounding gives $j_W(\kappa) = \kappa$. If f maps κ onto μ, then $j_W(f)$ maps κ onto $j_W(\mu)$, so $j_W(\mu) < \kappa^+$, contradiction. $\square$

We omit the proof of 0.10. It is like the proofs above, but uses Jackson's measure-bounding results for measures on projective ordnals.

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