Solutions to Final Exam. Discrete Mathematics 55

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Problem 1 (15pts). True or False? To discourage guessing, the problem will be graded as follows:

- 1 pt for each correct answer. • 0 pts for a blank. • -1 pts for each incorrect answer.
- If anything else but "True" or "False" is written, more than one answer is written, or the answer is hard to read, you will get -1 points.

(1) $(p \to q) \to (q \to p)$ is a tautology.

Answer: False. A statement $p \to q$ does not imply its converse $q \to p$. For example, when p is False and q is True, the whole statement reads: $(F \to T) \to (T \to F)$, which is the same as $T \to F$, i.e., F. Hence the given statement is not a tautology.

(2) If A is a proper subset of B, then $|A| < |B|$.

Answer: False. Counterexample: $A = \mathbb{Z}^+$ and $B = \mathbb{N}$. We have that $\mathbb{Z}^+ \subsetneq \mathbb{N}$, but both are infinitely countable and hence $|\mathbb{Z}^+| = |\mathbb{N}|$. (This question is from Midterm 1.)

(3) The ceiling function $\lceil\ \rceil : \mathbb{R} \to \mathbb{Z}$ is neither injective nor surjective.

Answer: False. The ceiling function is not injective, e.g., $\lceil 2.2 \rceil = 3 = \lceil 3 \rceil$, but it is surjective onto $\mathbb{Z}$ since $\lceil n \rceil = n$ for any integer n. (This question is from Midterm 1.)

(4) If p is prime, then $(p-1)!$ has a multiplicative inverse modulo p.

Answer: True. When p is prime, $(p-1)! = 1 \cdot 2 \cdot 3 \cdots (p-2) \cdot (p-1)$ is relatively prime with p since it is a product of positive integers smaller than p and hence individually relatively prime with p. But if n and p are relatively prime, then n has a multiplicative inverse modulo p, i.e., there is an integer m in $\mathbb{Z}_p$ such that $nm = 1$ in $\mathbb{Z}_p$, i.e., $nm \equiv 1 (\mathrm{mod}\ p)$. Thus, $n = (p-1)!$ has a multiplicative inverse modulo p and is, therefore, also called a *unit* modulo p. (In fact, Wilson's Theorem states that $(p-1)! \equiv -1 (\mathrm{mod}\ p)$. In our situation this means that $(p-1)!$ is its own multiplicative inverse modulo p because $(-1) \cdot (-1) = 1$ modulo any integer.)

(5) Mathematical Induction and Strong Mathematical Induction are logically equivalent.

Answer: True. Indeed, each types of induction is equivalent to the Well-Ordering Property. (This question is from Midterm 2.)

(6) The recursive definition $f(n) = f(n+1) + 7$ for $n \geq 1$ and $f(1) = 3$ produces a well-defined function on the set of positive integers.

Answer: True. When we solve for $f(n+1) = f(n) - 7$ we get a formula for defining $f(n+1)$ from $f(n)$. As there is a base case $f(1) = 3$, the function $f : \mathbb{Z}^+ \to \mathbb{R}$ is uniquely defined.

(7) If among any 26 people at a huge New Year's party (with all the people in the world) there are either 4 mutual friends or 5 mutual enemies, then the Ramsey number $R(4,5) = 26$.

Answer: False. The party may miss some particular (simple) subgraph of 26 people that has neither 4 mutual friends nor 5 mutual enemies, which, in turn, would imply that the Ramsey number $R(4,5)$ is > 26. On the other hand, even if all possible (simple) subgraphs of 26 people are present at the New Year's party, this does not preclude the possibility that the Ramsey number $R(4,5)$ could be smaller than 26. Finally, note the phrasing "either 4 mutual friends or 5 mutual enemies": if this is taken with the everyday language usage to mean an "exclusive OR," the situation does not conform to the definition of Ramsey number. Thus, the implication in the statement is false and we cannot conclude that $R(4,5) = 26$ based on the evidence. (In fact, it has been shown that $R(4,5) = 25$.)

(8) For any integer $n \geq 1$, the number of lattice paths (paths along the integer grid) from (0,0) to (n,n) that consist of moves only to the right and up and that do NOT go above the line $y = x$ is $\frac{1}{n}\binom{2n}{n}$.

Answer: False. These lattice paths are counted by the Catalan numbers, which are calculated by the formula $c_n = \frac{1}{n+1}\binom{2n}{n}$. For example, for $n = 1$ there is exactly one lattice path of the desired type from $(0, 0)$ to $(1, 1)$, confirmed by $\frac{1}{1+1}\binom{2}{1} = 1$; yet, the given (wrong) formula yields $\frac{1}{1}\binom{2}{1} = 2$.

(9) Two disjoint events are necessarily independent.

Answer: False. To the contrary, two disjoint events E and F are almost *never* independent: since $E \cap F = \emptyset$, $p(E \cap F) = 0 \neq p(E)p(F)$, unless $p(E) = 0$ or $p(F) = 0$. (This question is from Midterm 2.)

(10) For any random variable X, its variance $V(X)$ equals $E((X - \mu)^2)$ where $\mu = E(X)$.

Answer: True. By definition, variance measures the average "distance" between the random variable and its own average. Thus, the variance is the expected value of the square of the difference between the random variable and its own expected value $E(X)$.

(11) There is a graph consisting of a single vertex whose degree 6, but there is no graph consisting of a single vertex whose degree is 7.

Answer: True. A graph with 1 vertex v and three loops coming and going into v makes $\deg v = 2 \cdot 3 = 6$. However, by the corollary to the Handshake Theorem, the number of odd-degree vertices in a graph is always even; thus, having one vertex of odd degree ($= 7$) is impossible.

(12) The degree sequence completely determines an undirected graph.

Answer: False. For example, there are two different undirected graphs, each with two vertices v_1 and v_2 and degree sequences $\{2, 2\}$; namely, in the first graph make a loop at v_1 and another loop at v_2 while in the other graph join v_1 and v_2 via 2 edges. As an example with simple graphs, take one of your graphs to be the disjoint union of two cycles C_3 (two "triangles") and take your other graph to be the cycle C_6 (a "hexagon"): both have 6 vertices and same degree sequences $\{2, 2, 2, 2, 2, 2\}$ but the two graphs are definitely different (e.g., one is not connected while the other is).

(13) If an undirected simple graph has an Eulerian cycle, then it has a Hamiltonian cycle.

Answer: False. Take two cycles C_3 (two "triangles") and join them at one of their vertices making a figure that looks like a "bow tie" where 4 vertices have degrees 2 and one ("central") vertex has degree 4. Starting at the central vertex, traverse the first and then the second triangle, thereby making an Eulerian cycle. However, there is no way to make a Hamiltonian cycle because such a "cycle" will necessarily go through the central vertex twice.

(14) Regardless of how 11 ponies and 11 kids are standing in a field, as long as no two of them are on top of each other and no three are in a line, we can assign a different pony to each kid and let the kids run towards their assigned ponies so that no two kids' paths will intersect.

Answer: True. We did this problem in class using the interpretation of n boys and n girls being paired up for a dance so that when the boys walk to their respective girls the boys' paths do not intersect. A construction with a monovariant was the key to solving this problem. The complete matching produced from the set of boys to the set of girls is a "stronger" matching than just a complete matching because it has the extra property of non-intersection of the edges of the matching when the vertices of the graph are drawn and fixed in the plane.

(15) Although we found a formula for the number of ways to split m distinguishable kids into n groups, there is no known simple closed formula for this same number when the kids are all identical twins.

Answer: True. Using PIE, we found a summation formula involving the Stirling numbers of the second kind $S(m, j)$ for the number of ways to split m distinguishable kids into n groups. Although one can represent the question of number of ways to split m indistinguishable kids into n groups as a *partition* of m as a sum of at most n non-negative integers (where the order does not matter), it is true that there is no known simple closed formula to answer this question.

Problem 2 (12 pts) Find all solutions of the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2} + 8n$ with initial conditions $a_0 = 0$ and $a_1 = 2$. Explain and include all calculations.

Solution: The characteristic equation of the associated homogeneous linear recurrence relation (RR) $a_n = 6a_{n-1} - 9a_{n-2}$ is $x^2 - 6x + 9 = 0$. The latter factors as $(x-3)^2 = 0$, i.e., it has a double root $x_1 = x_2 = 3$. Thus, the solutions to the hom. RR are $b_n = \alpha 3^n + \beta\, n\, 3^n$ for some α and β.

Now let's find a particular solution to the original non-homogeneous RR $a_n = 6a_{n-1} - 9a_{n-2} + 8n$. Our guess will be a polynomial in n of degree 1 since the base 1 appearing in $8n \cdot (1)^n$ is not a root of the characteristic equation, i.e., let $p_n = A + Bn$ for some unknown A and B. We substitute p_n into the non-hom. RR:

$$\begin{aligned} A + Bn &= 6(A + B(n-1)) - 9(A + B(n-2)) + 8n \\ \Leftrightarrow\ A + Bn &= -3A - 3Bn + 12B + 8n \\ \Leftrightarrow\ 0 &= 4A - 12B + (4B - 8)n \end{aligned}$$

Equating the coefficients yields: $4A = 12B$ and $4B = 8$, i.e., $B = 2$ and $A = 6$, and $p_n = 6 + 2n$. Thus, all solutions to the non-hom. RR are $a_n = b_n + p_n = \alpha 3^n + \beta\, n\, 3^n + 6 + 2n$.

It remains to match the initial conditions: $a_0 = \alpha + 6 = 0$, i.e., $\alpha = -6$, and $a_1 = 3\alpha + 3\beta + 8 = 2$, i.e., $3\beta = 18 + 2 - 8 = 12$ and $\beta = 4$. Thus, the unique solution to the given RR is

$$a_n = -6 \cdot 3^n + 4\, n\, 3^n + 6 + 2n = (4n - 6)3^n + 2n + 6 \text{ for all } n \geq 0. \quad \square$$

Problem 3 (15 pts) Suppose you have a bag containing 10 red balls, 5 yellow balls, and 2 green balls. How many possible outcomes are there if:

(a) You take 10 balls from the bag, but every time you pick a ball you put it back into the bag before picking another ball. The order of drawing the balls does not matter, i.e., YYRGGGGGRG and YRYGGGGGRG are the same outcomes. Explain and include all calculations. (*Hint:* Since you are allowed to replace the picked balls, how many green balls at most can you eventually draw?) (6 pts)

Solution: Replacing the balls back into the bag means that we have an unlimited number of balls from each color. Thus, we have to choose 10 balls with colors R, Y, or G. The problem can be rephrased as solving the equation $x_1 + x_2 + x_3 = 10$ in non-negative integers x_1, x_2, and x_3, or equivalently distributing 10 identical biscuits among 3 distinguishable dogs (the "red," "yellow," and "green" dogs). The formula counting this is:

$$\binom{10+3-1}{3-1} = \binom{12}{2} = \frac{12 \cdot 11}{2} = 66. \quad \square$$

(b) You take 10 balls from the bag, but you do NOT replace the balls in the bag. The order of drawing the balls does not matter, i.e., YYRGGRRRRR and YRYGGRRRRR are the same outcomes. Explain and include all calculations. (*Hint:* There are many solutions, e.g., using generating functions, PIE, brute force, thinking of dogs and biscuits, etc.) (9 pts)

Solution: We have to draw 10 balls with colors R, Y, or G so that at most 10 red, at most 5 yellow, and at most 2 green balls are drawn. The problem can be rephrased as solving the equation $x_1 + x_2 + x_3 = 10$ in non-negative integers x_1, x_2, and x_3 such that $x_1 \leq 10$, $x_2 \leq 5$, and $x_3 \leq 2$, or equivalently distributing 10 identical biscuits among 3 distinguishable dogs (the "red," "yellow," and "green" dogs) so that the first dog gets at most 10 biscuits (this does not impose any restriction since we have 10 "biscuits" anyway), the second dog gets at most 5 biscuits, and the third dog gets at most 2 biscuits. Using generating functions, we are looking for the coefficient A of x^{10} in

$$(1+x+x^2+x^3+\cdots+x^{10})(1+x+x^2+x^3+x^4+x^5)(1+x+x^2)=\cdots+Ax^{10}+\cdots$$

Using brute force, there are three cases to consider.

- If 1 is chosen from the third factor (no green balls), then we can pick $x^{10-i}x^i$ from the first two factors where $i=0,1,2,3,4,5$ (at most 5 yellow balls). These are 6 possibilities.
- If x is chosen from the third factor (1 green ball), then we can pick $x^{9-i}x^i$ from the first two factors where $i=0,1,2,3,4,5$ (at most 5 yellow balls). These are 6 possibilities.
- If x^2 is chosen from the third factor (2 green balls), then we can pick $x^{8-i}x^i$ from the first two factors where $i=0,1,2,3,4,5$ (at most 5 yellow balls). These are 6 possibilities.

Thus, every possibility incurs 6 choices, i.e, a total of 18 outcomes. □

To solve this using PIE, suppose we have an unlimited number of balls in each color, and let A_1 be the subset of all possible outcomes with at least 6 yellow balls, and let A_2 be the subset of all possible outcomes with at least 3 green balls. Note that part (a) gives the number 66 of all outcomes of drawing 10 balls when no restrictions are placed on the number of balls in the various colors. What we want are the outcomes in the complement of A_1 and in the complement of A_2, i.e.,

$$|\overline{A_1\cup A_2}|=|S|-|A_1\cup A_2|\overset{\text{PIE}}{=}|S|-(|A_1|+|A_2|-|A_1\cap A_2|).$$

Note that to draw at least 6 yellow balls means that we can first draw 6 yellow balls, and then choose the remaining 4 balls without any color restrictions; but this is the same as distributing 4 biscuits to 3 dogs, i.e., as in part (a), $\binom{4+3-1}{3-1}=\binom{6}{2}=\frac{6\cdot5}{2}=15$. Similarly, to draw at least 3 green balls can be thought of drawing 3 green balls and then drawing 7 more balls without color restrictions, i.e., $\binom{7+3-1}{3-1}=\binom{9}{2}=\frac{9\cdot8}{2}=36$. Finally, we will need to calculate the size of the intersection $|A_1\cap A_2|$: we draw 6 yellow balls and 3 green balls and we need to choose the remaining 1 ball from the three colors, which yields, of course, 3 choices. Summarizing:

$$|\overline{A_1\cup A_2}|=66-(15+36-3)=18. \quad \square$$

Problem 4 (15 pts) Consider the number $N=2015^{2015}$.

(a) What is the remainder of N when it is divided by 4? What about when it is divided by 11? Explain and include all calculations. (8 pts)

Solution: Since $2015\equiv-1 (\text{mod } 4)$, we have $2015^{2015}\equiv(-1)^{2015}=-1\equiv3 (\text{mod } 4)$. Thus, the remainder of N when divided by 4 is 3. □

Now, $2015\equiv2 (\text{mod } 11)$, and 2 is relatively prime to 11. Thus, by Fermat's Little Theorem, $2^{10}\equiv1 (\text{mod } 11)$. Hence, $2015^{2015}\equiv2^{2015}=2^{201\cdot10+5}=\left(2^{10}\right)^{201}\cdot2^5\equiv1^{201}\cdot2^5=32\equiv10 (\text{mod } 11)$. Thus, the remainder of N when divided by 11 is 10. □

(b) What is the remainder of N is when it is divided by 44? Show all calculations and explain carefully. (*Hint:* You could use part (a) and the Chinese Remainder Theorem.) (7 pts)

Solution: N is a solution to the system of congruences:

$$\left|\begin{array}{ll} x\equiv3 & (\text{mod } 4)\\ x\equiv10 & (\text{mod } 11).\end{array}\right.$$

Since 4 and 11 are relatively prime, by the Chinese Remainder Theorem there is a unique solution to this system modulo 44. Thus, if we find this unique solution among the remainders modulo 44, we will have found the remainder of N modulo 44. One slick way to find this solution is to note that $x\equiv3\equiv-1 (\text{mod } 4)$ and $x\equiv10\equiv-1 (\text{mod } 11)$, so if we find a remainder congruent to -1

modulo 44, that number will be also congruent to -1 modulo 4 and modulo 11. But 43 is precisely such a number: $43 \equiv -1 \pmod{44}$, so the wanted remainder of N modulo 44 is 43. $\square$

Another way to locate 43 is simply to exhaustively go through all numbers from 0 to 43 that are congruent to 10 modulo 11, i.e., 10, 21, 32, and 43, and see which is congruent to 3 modulo 4. The corresponding remainders modulo 4 are 2, 1, 0, and 3, i.e., we locate once again 43 as our answer. $\square$

Problem 5 (15 pts) Determine whether each of the following pairs of random variables are independent or not when two fair dice are tossed. Explain and include all calculations.

(a) X is the sum of values on the two dice and Y is the value of the first die. For example, if the two die show 5 and 2, then $X = 7$ and $Y = 5$. (6 pts)

Solution: For independence of X and Y, we check if $p(X = k \text{ and } Y = m) \stackrel{?}{=} p(X = k) \cdot p(Y = m)$ for every possible values k of X and m of Y. For example, when $k = 12$ and $m = 1$, we have:

- $p(X = 12) = 1/36$ since there are $6 \cdot 6 = 36$ total outcomes from tossing two dice and only one of these outcomes gives the sum 12, namely, both die need to show 6.
- $p(Y = 1) = 1/6$ since all of the 6 outcomes are equally likely on a fair die.
- $p(X = 12 \text{ and } Y = 1) = 0$ since it can't be that the sum is 12 but the first die shows 1.

Thus, $p(X = 12 \text{ and } Y = 1) = 0 \neq p(X = 12) \cdot p(Y = 1) = \frac{1}{36} \cdot \frac{1}{6}$, and X and Y are dependent. $\square$

(b) X is the sum modulo 2 of values on the two dice and Y is the value of the first die. For example, if the two die show 5 and 2, then $X = 1$ and $Y = 5$. (9 pts)

Solution: This time we will calculate all probabilities and show independence of the two variables.

- $p(X = 0) = p(X = 1) = 1/2$ because there are as many outcomes when the sum is even and there are when the sum is odd. Indeed, $(even, even)$ and (odd, odd) will yield even sums, i.e., $3 \cdot 3 + 3 \cdot 3 = 18$ possibilities for even sums, which is exactly half of the total of 36 outcomes.
- $p(Y = m) = 1/6$ for each $m = 1, 2, 3, 4, 5, 6$ since this is a fair die.
- To have the first die show 2 and the sum to be even, we need the second number also to be even, i.e., there are 3 possibilities: (2,2), (2,4), and (2,6), so $p(X = 0 \text{ and } Y = 2) = 3/36 = 1/12$. Similarly, for any other even m: $p(X = 0 \text{ and } Y = m) = 1/12$.
- To have the first die show 1 and the sum to be even, we need the second number also to be odd, i.e., there are 3 possibilities: (1,1), (1,3), and (1,5), so $p(X = 0 \text{ and } Y = 1) = 3/36 = 1/12$. Similarly, for any other odd m: $p(X = 0 \text{ and } Y = m) = 1/12$.
- To have the first die show 2 and the sum to be odd, we need the second number to be even, i.e., there are 3 possibilities: (2,1), (2,3), and (2,5), so $p(X = 1 \text{ and } Y = 2) = 3/36 = 1/12$. Similarly, for any other even m: $p(X = 1 \text{ and } Y = m) = 1/12$.
- To have the first die show 1 and the sum to be odd, we need the second number to be even, i.e., there are 3 possibilities: (1,2), (1,4), and (1,6), so $p(X = 1 \text{ and } Y = 1) = 3/36 = 1/12$. Similarly, for any other odd m: $p(X = 1 \text{ and } Y = m) = 1/12$.

Summarizing, $p(X = k \text{ and } Y = m) = 1/12$, $p(X = k) = 1/2$, and $p(Y = m) = 1/6$ for all values $k = 0, 1$ and $m = 1, 2, 3, 4, 5, 6$. Hence

$$p(X = k \text{ and } Y = m) = \frac{1}{12} = \frac{1}{2} \cdot \frac{1}{6} = p(X = k) \cdot p(Y = m),$$

and the two variables X and Y are independent. $\square$

Problem 6 (13 pts) Recall that the Fibonacci numbers are defined by $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$, $f_0 = 0$ and $f_1 = 1$. Prove that for any $n \geq 1$: $f_0 f_1 + f_1 f_2 + \cdots + f_{2n-1} f_{2n} = f_{2n}^2$. (*Note:* There are at least two ways to solve this problem, one much easier than the other.)

Solution 1: The standard way to prove this identity is to use induction on n.

First we confirm the base case for $n = 1$: $f_0 f_1 + f_1 f_2 = 0 \cdot 1 + 1 \cdot 1 = 1^2 = f_2^2$.

Assume that the wanted statement is true for n. To show it for $n+1$, start from the lefthand side for $n+1$:

$$\begin{aligned} LHS_{n+1} &= (f_0 f_1 + f_1 f_2 + \cdots + f_{2n-1} f_{2n}) + f_{2n} f_{2n+1} + f_{2n+1} f_{2n+2} \\ &\overset{IH}{=} f_{2n}^2 + f_{2n} f_{2n+1} + f_{2n+1} f_{2n+2} = f_{2n}(f_{2n} + f_{2n+1}) + f_{2n+1} f_{2n+2} \\ &\overset{RR}{=} f_{2n} f_{2n+2} + f_{2n+1} f_{2n+2} = (f_{2n} + f_{2n+1}) f_{2n+2} \overset{RR}{=} f_{2n+2} f_{2n+2} = f_{2n+2}^2. \quad \square \end{aligned}$$

Solution 2 (sketch): As we have done in class, solve the RR to get $f_n = \frac{1}{\sqrt{5}} \left(\phi^n - \overline{\phi}^n \right)$ for $n \geq 0$, where $\phi = (1 + \sqrt{5})/2$ and $\phi = (1 - \sqrt{5})/2$ are the two roots of the quadratic equation $x^2 - x - 1 = 0$. As such, by Vieta's formulas, $\phi\overline{\phi} = -1$ and $\phi + \overline{\phi} = 1$. Substitute into the desired identity all f_j's and using the formula for a finite geometric sum and some algebraic manipulations, show that the two sides are equal. $\square$

Problem 7 (15 pts) King Arthur's n knights are invited to a New Year's performance in the King's court. As we know, some knights are friends with each other and some are enemies (the relationships of friendship and enmity are mutual). Suppose that for any two knights x and y, any other knight is a friend with x or y and there are at least 2 other knights with whom x and y are both friends.

(a) Write the conditions above (the third sentence) as a compound proposition. Clearly label and state the simple propositions (or propositional functions) you use in your solution. (5 pts)

Solution: Let $P(x, y)$ be the propositional function "Knights x and y are friends." Then $\neg P(x, y)$ reads as "Knights x and y are enemies." The group of n knights is the universe in this problem. Then the given conditions can be encoded as follows:

$$\forall x \forall y \Big(y \neq x \to \forall z \big(z \neq x \land z \neq y \to (P(x, z) \lor P(y, z)) \big) \\ \bigwedge \exists t, u \ \big(t \neq u \land t \neq x \land t \neq y \land u \neq x \land u \neq y \land P(t, x) \land P(t, y) \land P(u, x) \land P(u, y) \big) \Big). \quad \square$$

(b) Prove that there is a way for King Arthur to seat all n knights in one *row* at the New Year's performance so that everyone is sitting only next to friends. Explain carefully. (*Note:* The propositional rephrase in part (a) may NOT be helpful here. You should try something else.) (10 pts)

Solution 1: Let's look at the problem from the viewpoint of graph theory. The friendship graph with vertices all knights is simple, since any two knights are either connected by one edge only (when they are friends) or not connected at all (when they are enemies), and there are no loops. For any two (distinct) vertices x and y let us count the number $S_{x,y}$ of edges incident with x or y, i.e., $S_{x,y} = \deg x + \deg y$. Let t and u be some other two knights who are mutual friends with both x and y; then t and u contribute a total of $2 + 2 = 4$ to the sum $S_{x,y}$. Any of the remaining $n - 4$ vertices (not counting x, y, t, u) contributes to $S_{x,y}$ at least one edge, i.e., a total of at least $n - 4$ more edges. In total, $S_{x,y} \geq 4 + (n - 4) = n$. (Note that we didn't count here the possible edge between x and y.)

If our graph has only 1 vertex, then King Arthur can seat his only knight without trouble in a row. If our graph has at least 2 vertices, then it actually has at least 4 vertices (why?) and the hypothesis of Ore's Theorem is satisfied. Hence, there is an Hamiltonian cycle in our graph; i.e., King Arthur could have first seated the n knights at the Round Table so that any two adjacent knights are friends. Now the King only has to ignore one of the friendships of two knights sitting next to each other and straighten up the circular arrangement into a row for the New Year's show. $\square$

Solution 2: There is a simple constructive way to arrange the knights. We shall say that we have arranged m of the knights in a "friendly chain" of length m if these m knights are sitting in a row so that any two adjacent knights are friends.

Again, if there is only 1 knight, then we can seat him with no problem in a row. If there are two knights, then there are at least 2 more knights by hypothesis who are their mutual friends. At any rate, there is a pair (k_1, k_2) of two knights who are friends. Seat them next to each other, thereby creating a friendly chain of length 2. Now pick any other knight k_3; by hypothesis, k_3 is a friend of k_1 or k_2 (or both); seat k_3 next to whomever is a friend of his among k_1 and k_2 (or choose one of them if they are both his friends). Now we have a friendly chain of length 3. Suppose you can extend this to a friendly chain of length m for some $m \geq 3$. Then pick any other knight k_{m+1} who is not seated yet; by hypothesis, k_{m+1} is a friend of one (or both) of the knights sitting at the ends of the chain; seat k_{m+1} next to whomever is a friend of his among these two knights at the end of the chain (or choose one of them if they are both his friends). Now we have a friendly chain of length $m + 1$. By induction, we can continue the process of extending the friendly chain until we seat everyone down in a friendly chain of length n. $\square$

Note that Solution 2 used only part of the given hypothesis.