Homework 3 Solutions, Math 55

1.8.4. There are three cases: that a is minimal, that b is minimal, and that c is minimal. If a is minimal, then $a \leq b$ and $a \leq c$, so $a \leq \min\{b, c\}$, so then

$$\min\{a, \min\{b, c\}\} = a.$$

Also $a \leq b$, so $\min\{a, b\} = a$, and then

$$\min\{\min\{a, b\}, c\} = \min\{a, c\} = a$$

since $a \leq c$ also. Thus $\min\{a, \min\{b, c\}\} = \min\{\min\{a, b\}, c\}$. The other two cases are similar and I'm too lazy to write them out.

1.8.6. If x and y are integers of opposite parity, without loss of generality we can assume that x is even and y is odd. Then $x = 2a$ for some integer a and $y = 2b+1$ for some integer b, so

$$5x + 5y = 10a + 10b + 1 = 2(5a + 5b) + 1$$

is odd.

1.8.10. The integers $2 \cdot 10^{500} + 15$ and $2 \cdot 10^{500} + 16$ are consecutive positive integers, and in general consecutive positive integers can never both be perfect squares. Indeed, if a positive integer n is a perfect square, it is equal to a^2 for some positive integer a, and the next perfect square strictly greater than n must be

$$(a + 1)^2 = a^2 + 2a + 1 = n + 2a + 1 > n + 1$$

since $2a + 1 > 1$ when $a \geq 1$.

1.8.14. It need not be that a^b is rational whenever a and b are rational. For example, if $a = 2$ and $b = 1/2$, then $a^b = 2^{1/2} = \sqrt{2}$.

1.8.18. We start with existence. Let n be the largest integer less than or equal to r. In other words, we choose n so that $n \leq r < n + 1$. Since r is irrational, we know that $r \neq n$, so $n < r < n + 1$. If $|r - n| < 1/2$, then n is an integer whose distance to r is less than $1/2$, so we're done. So, suppose $|r - n| \geq 1/2$. Since $r > n$, we know that $r - n > 0$ so $|r - n| = r - n \geq 1/2$, which means that $n - r \leq 1/2$. But we also know that $n + 1 > r$, so $|(n + 1) - r| = (n + 1) - r$. Then

$$|(n + 1) - r| = (n + 1) - r = (n - r) + 1 \leq 1/2.$$

If $(n + 1) - r = 1/2$, then we would have $r = n + 1/2$, but r was supposed to be irrational, so that's a contradiction. So then $|(n + 1) - r| = (n + 1) - r < 1/2$, so $n + 1$ is an integer whose distance to r is less than $1/2$, so again we're done.

For uniqueness, suppose n and n' are distinct integers whose distance to r is less than $1/2$. Then

$$|n - n'| = |(n - r) - (n' - r)| \leq |n - r| + |n' - r| < \frac{1}{2} + \frac{1}{2} = 1,$$

using the triangle inequality. But the distance between two distinct integers is always greater than 1, so this is impossible.

1.8.22. If x is a nonzero real number, then $x - 1/x$ is also a real number, so $(x - 1/x)^2 \geq 0$ since the square of any real number is nonnegative. Expanding this out, we get $x^2 - 2x(1/x) + 1/x^2 \geq 0$. But the middle term is equal to 2, and then adding 2 to both sides gives

$$x^2 + 1/x^2 \geq 2.$$

1.8.30. Let $f(x, y) = 2x^2 + 5y^2$. First notice that (x, y) is a solution if and only if $(|x|, |y|)$ is a solution, so it suffices to look for solutions (x, y) where both x and y are nonnegative. If $x \geq 3$, then

$$f(x, y) \geq 2 \cdot 3^2 + 5y^2 \geq 2 \cdot 9 = 18 > 14$$

so any solution must have $x \geq 2$. Also if $y \geq 2$, we similarly have

$$f(x, y) \geq 5 \cdot 2^2 = 20 > 14$$

so we must have $y \leq 1$. There are only 6 pairs (x, y) where $0 \leq x \leq 2$ and $0 \leq y \leq 1$. We check all of these by hand.

$$\begin{aligned} f(0,0) &= 0 \\ f(0,1) &= 5 \\ f(1,0) &= 2 \\ f(1,1) &= 7 \\ f(2,0) &= 8 \\ f(2,1) &= 13 \end{aligned}$$

Since none of these are equal to 14, we conclude that $f(x, y) = 14$ has no integer solutions.

1.8.32. If m and n are arbitrary integers and $x = m^2 - n^2$, $y = 2mn$, and $z = m^2 + n^2$, then

$$\begin{aligned} x^2 + y^2 &= (m^2 - n^2)^2 + (2mn)^2 \\ &= m^4 - 2m^2n^2 + n^4 + 4m^2n^2 \\ &= m^4 + 2m^2n^2 + n^4 \\ &= (m^2 + n^2)^4 = z^2. \end{aligned}$$

Thus each pair of integers (m, n) gives rise to a triple of integers (x, y, z) that is a solution to $x^2+y^2 = z^2$. Now notice that fixing $n = 1$ gives us solutions of the form $(m^2-1, 2m, m^2+1)$. For positive integers $m \neq m'$, clearly $(m^2 - 1, 2m, m^2 + 1) \neq (m'^2, 2m', m'2 + 1)$, so there are infinitely many solutions (x, y, z) to $x^2 + y^2 = z^2$ in the positive integers.

1.8.34. Suppose $\sqrt[3]{2}$ were rational. Then we can write $\sqrt[3]{2} = p/q$ where p and q are integers and p/q is in lowest terms. Cubing both sides and clearing denominators, we get $p^3 = 2q^3$. Thus p^3 is even. If p were odd, it would be equal to $2k + 1$ for some integer k, and then

$$p^3 = (2k + 1)^3 = 2(4k^3 + 6k^2 + 3k) + 1$$

which is odd. But we know that p^3 is even, so p cannot be odd, so it must be even. Thus $p = 2k$ for some integer k. Then

$$2q^3 = p^3 = (2k)^3 = 8k^3$$

which means that $q^3 = 4k^3 = 2(2k^3)$. This means that q^3 is even, so as above, q must also be even. But p and q were supposed to be in lowest terms, so this is a contradiction.

1.8.36. Let a be rational and x irrational. We will show that $y = (a + x)/2$ is irrational and y is between a and x. First, to see that y must be irrational. If it were rational, then we would have

$$x = 2y - a$$

but then the left hand side is clearly rational, whereas x was supposed to be irrational. Thus y must be irrational.

Next, we need to show that y is between a and x. Since a is rational and x is irrational, we know that $a \neq x$, so either $a < x$ or $x < a$. If $a < x$, then

$$a + a < a + x < x + x$$

so, dividing by 2, we get

$$a = \frac{a+a}{2} < \frac{a+x}{2} < \frac{x+x}{2} = x.$$

Thus $a < y < x$. If $x < a$, one proves that $x < y < a$ in an analogous way.

1.8.42. You can tile a standard checkerboard with all four corners removed using dominoes. Each row has an even number of squares (either 6 or 8), so you can just place the dominoes end-to-end in each row. If I were doing this by hand, I'd draw a picture, but I'm not, so you'll just have to imagine it.

1.8.44. Suppose the original 5×5 checkerboard (without the corners removed) is colored with alternating white and black squares, starting with a white square in the top left corner. Then there are 3 rows with 3 white squares and 2 rows with 2 white squares, so a total of 13 white squares. Then there are 25 total squares, so the remaining 12 squares are all black. Now suppose we remove 3 of the corner squares. All of the corner squares are white, so we wind up with 10 white squares and 12 black squares. Each domino needs to cover a black square and a white square, but there are a different number of white squares and black squares left, so it is impossible to tile this checkerboard with dominoes.

2.1.10. I don't know how to write down an explanation for any of these without saying "duh," so here are the answers.

(a) True.

(b) True.

(c) False.

(d) True.

(e) False.

(f) False.

(g) True.

2.1.12. I'm on a computer, so I can't draw. Sorry.

2.1.16. Again, I'm on a computer, so I can't draw.

2.1.18. Let $A = \emptyset$ and $B = \{\emptyset\}$. Then clearly $\emptyset \in \{\emptyset\}$ and $\emptyset \subset \{\emptyset\}$.

2.1.20. Again, I don't know how to write down an explanation for any of these without saying "duh," so here are the answers.

(a) 0.

(b) 1.

(c) 2.

(d) 3.

2.1.22.

(a) The empty set can never be the power set of a set S, since any set S must have at least $\emptyset$ as a subset, so at least $\emptyset$ must be an element of the power set, but $\emptyset \notin \emptyset$.

(b) This is the power set of $\{a\}$.

(c) This cannot be the power set of any set. If it were the power set of some set S, then we would have $\{a, \emptyset\} \subseteq S$, so $\emptyset \in S$, so then $\{\emptyset\}$ would also be a subset of S. But $\{\emptyset\}$ is not an element of the set described in the problem.

(d) This is the power set of $\{a, b\}$.

2.1.26. Suppose $(a, b) \in A \times B$. Then $a \in A$ and $b \in B$. But $A \subseteq C$ and $B \subseteq D$, so $a \in C$ and $b \in D$. So then $(a, b) \in C \times D$. Thus $A \times B \subseteq C \times D$.

2.1.32.

(b) There should be $3 \times 2 \times 2 = 12$ elements.

$$\begin{aligned} C \times B \times A = \{&(0, x, a), (0, x, b), (0, x, c), \\ &(0, y, a), (0, y, b), (0, y, c), \\ &(1, x, a), (1, x, b), (1, x, c), \\ &(1, y, a), (1, y, b), (1, y, c)\} \end{aligned}$$

(d) There should be 8 elements. I'm too lazy to list them all.

2.1.38. If A and B are nonempty and $A \neq B$, then either there exists an element $a \in A$ such that $a \notin B$, or else there exists an element $b \in B$ such that $b \notin A$. Without loss of generality, assume there exists $a \in A$ such that $a \notin B$. Since B is nonempty, there exists some $b \in B$ (which may or may not be in A, but it doesn't matter either way). Then $(a, b) \in A \times B$, but $(a, b) \notin B \times A$ since $a \notin B$. Thus $A \times B \neq B \times A$.

2.2.2.

(a) $A \cap B$.

(b) $A - B$.

(c) $A \cup B$.

(d) $\overline{A} \cup \overline{B}$ or $\overline{A \cap B}$.

2.2.4.

(a) Since $A \subseteq B$, we have $A \cup B = B$.

(b) Since $A \subseteq B$, we have $A \cap B = A$.

(c) $A \setminus B = \{f, g, h\}$.

(d) $B \setminus A = \emptyset$.

2.2.12. First, note that if $x \in A$, then clearly $x \in A \cup (A \cap B)$, so we definitely have $A \subseteq A \cup (A \cap B)$. Conversely, if $x \in A \cup (A \cap B)$, then either $x \in A$, or else $x \in A \cap B$. In either case, we see that we must have $x \in A$, which shows that $A \cup (A \cap B) \subseteq A$. Thus $A = A \cup (A \cap B)$.

2.2.14. If $A = \{1, 3, 5, 6, 7, 8, 9\}$ and $B = \{2, 3, 6, 9, 10\}$, then one can verify that $A - B$, $B - A$ and $A \cap B$ are all as specified in the problem.

2.2.16(d). If $x \in A \cap (B \setminus A)$, then $x \in A$ and $x \in B \setminus A$. This means that $x \in A$, and $x \in B$ but $x \notin A$. So we have $x \in A$ and $x \notin A$, so this is a contradiction. Thus $A \cap (B \setminus A)$ has no elements.

2.2.18(c). If $x \in (A - B) - C$, then $x \in A - B$ and $x \notin C$. This means that $x \in A$, $x \notin B$ and $x \notin C$. So $x \in A$ and $x \notin C$, so $x \in A - C$. Thus $(A - B) - C \subseteq A - C$.

2.2.24. If $x \in (A - C) - (B - C)$, then $x \in A - C$ and $x \notin B - C$. Since $x \in A - C$, we have $x \in A$ and $x \notin C$. Since $x \notin B$, we know that either $x \notin B$ or else $x \in C$. But we already know that $x \notin C$, so actually we must have $x \notin B$. Thus $x \in A$ and $x \notin B$, so $x \in A - B$, but then also $x \notin C$ so $x \in (A - B) - C$. This shows that $(A - C) - (B - C) \subseteq (A - B) - C$. The proof of the reverse inclusion is similar.

2.2.26(b). On a computer, can't draw.

2.2.30.

(a) Nope. For example, take $A = \emptyset$, $B = \{1\}$, and $C = \{1\}$. Then $A \neq B$ but $A \cup C = B \cup C$.

(b) Nope. For example, take $A = \{1\}$, $B = \{2\}$ and $C = \emptyset$. Then $A \cap C = B \cap C$ but $A \neq B$.

(c) Yes. Suppose $x \in A$. Then $x \in A \cup C$, and $A \cup C = B \cup C$, so $x \in B \cup C$ so either $x \in B$ or $x \in C$. If $x \in B$, we're done, so suppose $x \notin B$. Then we must have $x \in C$. Then $x \in A \cap C$, but $A \cap C = B \cap C$, so $x \in B \cap C$. But this is a contradiction, since we assumed that $x \notin B$ but $B \cap C \subseteq B$. Thus we have just shown that $A \subseteq B$. The proof of the reverse inclusion is identical.

2.2.44. Let $n = \max\{|A|, |B|\}$. Then $|A \cup B| \leq n + n = 2n$, so $A \cup B$ is finite.