HOMEWORK NO.5

8.5.2

(a) The probability that it takes you at most 15 min. to get to your 8 a.m. section is $\int_0^{15} f(t)dt$.
(b) The probability that is takes you more than thirty minutes is $\int_{30}^{\infty} f(t)dt$.

8.5.4

(a) We require that f be non-negative on the interval $[0,1]$. Accordingly, $k \geq 0$. Moreover, the definitions the definition demands that

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^1 kx^2(x-1)dx = k\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = \frac{k}{12} = 1$$

whence, $k = 12$.

(b) As per part (a), let $k = 12$. Then

$$P(X \geq \frac{1}{2}) = \int_{\frac{1}{2}}^{\infty} f(x)dx = \int_{\frac{1}{2}}^{1} 12x^2(1-x)dx = \int_{\frac{1}{2}}^{1}(12x^2 - 12x^3)dx = \frac{11}{16}$$

(c) Applying the definition $\mu = \int_{-\infty}^{\infty} xf(x)dx$, gives

$$\mu = \int_0^1 x.12x^2(1-x)dx = 12\int_0^1 (x^3 - x^4)dx = \frac{3}{5}$$

8.5.6

(a) It is immediate that f is non-negative. The area under the curve is simply that of a triangle having base length 9 and height .2, i.e.... oh wait, I am confused. As illsutrated in the text, the triangle given by the graph of f has area .9, precluding f from being a probability function at all. Taking some poetic license, let us assume that the base of the triangle begins at the origin, giving it length 10. Then $A = \frac{1}{2}(10)(.2) = 1$ as required.

(b) With the assumptions of (a) in place (also assume that $f(3) = .1$ as below) we have $P(X < 3) = \frac{1}{2}(3)(.1) = .15$.

(c) f is piecewise linear, and is given by

$$f(x) = \begin{cases} \frac{1}{30}x & \text{if } 0 \leq x < 6 \\ -\frac{1}{20}x + \frac{1}{2} & \text{if } 6 \leq x < 10 \\ 0 & \text{otherwise} \end{cases}$$

The mean is thus

$$\mu = \int_{-\infty}^{\infty} xf(x)dx = \int_0^6 x\left(\frac{1}{30}x\right)dx + \int_6^{10} x - \left(\frac{1}{20}x + \frac{1}{2}\right)dx$$

8.5.8

(a) As explained in example 3, p. 578 of STEWART ,

$$\mu = 1000 \Rightarrow f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{1000}e^{\frac{-t}{1000}} & \text{if } t \geq 0 \end{cases}$$

(i)

$$P(0 \leq X \leq 200) = \int_0^{200} \frac{1}{1000}e^{-t}1000dt = -e^{-1/5} + 1$$

(ii)

$$P(X > 800) = \int_{800}^{\infty} \frac{1}{1000}e^{-t}1000dt = \lim_{x\to\infty}\left[-e^{-t/1000}\right]_{800}^{\infty} = 0 + e^{-4/5} = e^{-4/5}$$

(b) We want to find m satisfying $\int_m^{\infty} f(t)dt = \frac{1}{2}$. This gives the following string of implications.

$$\lim_{x\to\infty}\int_m^x f(t)dt = \frac{1}{2} \Rightarrow \lim_{x\to\infty}\left[-e^{-t/1000}\right]_m^x = \frac{1}{2} \Rightarrow 0 + e^{-m/1000} = \frac{1}{2} \Rightarrow m = -1000\ln\frac{1}{2} \text{ hrs.}$$

8.5.10

(a) Stewart tells us that $\mu = 69$ and $\sigma = 2.8$. Accordingly,

$$P(65 \leq X \leq 73) = \int_{65}^{73} \frac{1}{2.8\sqrt{2\pi}}e^{-\frac{(x-69)^2}{2\cdot 2.8^2}}dt$$

I have omitted the details of my scratch work but— applying Simpson's rule by hand with $n = 1500$— I found that

$$\int_{65}^{73} \frac{1}{2.8\sqrt{2\pi}}e^{-\frac{(x-69)^2}{2\cdot 2.8^2}}dt \approx 0.847$$

(b)
$$P(X > 72) = 1 - P(0 \leq X \leq 72) \approx 1 - .858 = .142$$
Thus 14.2% of the population is more than 6 feet tall.

8.5.12

(a)
$$P(0 \leq X \leq 480) = \int_0^{480} \frac{1}{12\sqrt{2\pi}} e^{-\frac{(x-9.4)^2}{2 \cdot 4.2^2}} \approx .0478$$
So there is approximately a 4.78% chance that a particular box contains less than 480 g of cereal.

(b) We want to find μ so that $P(0 \leq X < 500) = 0.05$ Using an abacus to find $P(0 \leq X < 500)$ for various values of μ , we find that if $\mu = 519.73$, $P = .05007$; and if $\mu = 519.74$, $P = .04998$. So a reasonable target is 519.74 g.

8.R.20
$$P(250 \leq X \leq 280) = \int_{250}^{280} \frac{1}{15\sqrt{2\pi}} e^{-\frac{(x-268)^2}{2 \cdot 15^2}} dx \approx .673$$
So the percentage of pregnancies that last between 250 and 280 days is about 67.3%

8.R.21

(a) The probability density function is
$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{8} e^{\frac{-t}{8}} & \text{if } t \geq 0 \end{cases}$$
Thus
$$P(0 \leq X \leq 3) = \int_0^3 \frac{1}{8} e^{\frac{-t}{8}} dt = -e^{-\frac{t}{8}} \Big|_0^3 = -e^{\frac{-3}{8}} + 1$$

(b)
$$P(X > 10) = \int_{10}^{\infty} \frac{1}{8} e^{\frac{-t}{8}} dt = \lim_{x \to \infty} \left[-e^{\frac{-t}{8}} \right]_{10}^{x}$$

(c) We need to find m so that $P(X \geq M) = \frac{1}{2}$. That is,
$$\int_m^{\infty} \frac{1}{8} e^{\frac{-t}{8}} dt = \frac{1}{2} \Rightarrow \lim_{x \to \infty} \left[-e^{-\frac{t}{8}} \right]_m^x \Rightarrow e^{\frac{-m}{8}} = \frac{1}{2} \Rightarrow m = -8 \ln \frac{1}{2} \text{ minutes}$$

11.1.4 The sequence is $\langle 1, \frac{3}{5}, \frac{1}{2}, \frac{1}{8}, \ldots \rangle$

11.1.6 $a_n = 2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)$, giving the sequence $\langle 2, 8, 48, 384, 3840, \ldots \rangle$.

11.1.8 This sequence is defined recursively, and its first few terms are $\langle 4, \frac{4}{3}, 4, \ldots \rangle$. Note that since "4" occurs twice in the part of the sequence we have calculated explicitly , we have in fact described it in its entirety: once we encounter 4, the recursive definition "thinks" it is back at a_1.

11.1.10 $a_n = \frac{1}{2n}$

11.1.12 $a_n = \frac{(-1)^n n}{(n+1)^2}$

11.1.14 $a_n = 3 + (-1)^{n+1} \cdot 2$

11.1.16 This sequence converges. Note that
$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n+1}{3n-1} = \lim_{n \to \infty} \frac{1 + \frac{1}{n}}{3 - \frac{1}{n}} = \frac{1}{3}$$

11.1.18
$$a_n = \frac{\sqrt{n}}{1 + \sqrt{n}} = \frac{1}{\frac{1}{\sqrt{n}} + 1}$$
Since $\frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$, $a_n \to 1$.

11.1.20 Note that $a_n = \frac{\sqrt{n}}{\frac{1}{\sqrt{n}+1}}$. The denominator tends to 1 as $n \to \infty$ while $\sqrt{n}$ grows without bound. Thus $a_n \to \infty$.

11.2.22 Note that for even n
$$a_n = \frac{n^3}{n^3 + 2n^2 + 1} = \frac{1}{1 + \frac{2}{n} + \frac{1}{n^3}} \to 1 \text{ as} n \to \infty$$

For odd n, however

$$a_n = \frac{n^3}{n^3 + 2n^2 + 1} = \frac{-1}{1 + \frac{2}{n} + \frac{1}{n^3}} \to -1 \text{ as} n \to \infty$$

From these facts we can infer that a_n admits of two convergent subsequences whose respective limits disagree. Consequently,a_n itself fails to have a limit.

11.1.24 As $n \to \infty$, $\frac{2}{n} \to 0$. Since $\cos(0) = 1$, $\lim_{n\to\infty} a_n = 1$.

11.1.26 $2n \to \infty$ as $n \to \infty$. Since $\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$, it follows that $\lim_{n\to\infty} a_n = \frac{\pi}{2}$.

11.1.28

$$a_n = \frac{\ln n}{\ln 2n} = \frac{\ln n}{\ln 2 + \ln n} = \frac{1}{\frac{\ln 2}{\ln n} + 1} \to \frac{1}{0+1} = 1 \text{ as} n \to \infty$$

11.1.30 $a_n = n \cos n\pi = n(-1)^n$. The last equality follows from the periodicity of the cosine function. Considering only the even terms, we see that a_n diverges.

11.1.42 Without a calculator it is straightforward to check that the limit is 2, since $2/\pi < 1$, whence $(2/\pi)^n$ tends to 0 as n approaches infinity.

11.1.44 Note that

$$0 \leq \frac{|\sin n|}{\sqrt{n}} \leq \frac{1}{\sqrt{n}}$$

Thus, by the squeeze theorem and theorem 6 $a_n = \frac{\sin n}{\sqrt{n}}$ tends toward 0 as $n \to \infty$.