Review Topics for Midterm 1 in Calculus 1B

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1. Definitions

Be able to **write** precise definitions for any of the following concepts (where appropriate: both in words and in symbols), to **give** examples of each definition, to **prove** that these definitions are satisfied in specific examples. Wherever appropriate, be able to **graph** examples for each definition. What is

(1) "$du = u'(x)dx$"? Is this a formula? What does it mean and where is it used?
(2) "udv" and "vdu"? Are these formulas? What are they used for?
(3) *trigonometric substitution*? When is it used? Geometric interpretation. Table of trig. substitution.
(4) a *trigonometric integral*? Examples. Which type of trigonometric integrals have we learned to solve?
(5) a *rational function*? What does it mean to represent it in terms of partial fractions?
(6) *long division*? When do we need to use long division?
(7) an *irreducible quadratic factor*? How do we complete the square in such a quadratic polynomial?
(8) a *rationalizing substitution*? When do we use it?
(9) *rationalizing numerator or denominator*? Is this the same as a rationalizing substitution?
(10) a *Riemann sum* R_n, L_n or M_n? An *approximation* T_n or S_n? How are these sums related to one another? Which are expected to be more precise than the others? What is an *error bound* for a sum and how do we use it? What is the difference between an approximation, the error of the approximation, and an estimate for the error of the approximation?
(11) an *improper integral*? How many types of improper integrals are there? When is an improper integral *convergent* and when is it *divergent*? How can we compare two improper integrals and use knowledge of convergence/divergence for one of them to conclude convergence/divergence for the other? Why do we care if an improper integral converges or diverges?
(12) an *indeterminacy*? How many different types of indeterminacies are there? When do indeterminacies arise? How are limits and L'Hospital's rule related to indeterminacies?
(13) the *arc length* of a curve? the *surface area of a solid of revolution*? How do we compute them? What conditions must $f(x)$ satisfy so that we can apply the arc length formula? What if $f'(x)$ is not continuous at one end of the interval $[a, b]$: how do we deal with such a problem?
(14) a *continuous random variable* and a *probability density function*? How do you know if a function is a probability density function?
(15) the *mean* of a probability density function and how do we find it?
(16) the *median* of a probability density function and how do we find it? What is the difference between the mean and the median?
(17) an *exponential density function*? a *normal distribution*? How do they differ?
(18) **(bonus)** the *standard deviation* of a random variable with a probability function $f(x)$?

2. Theorems

Be able to **write** what each of the following theorems (laws, propositions, corollaries, etc.) says. Be sure to understand, distinguish and **state** the conditions (hypothesis) of each theorem and its conclusion. Be prepared to **give** examples for each theorem, and most importantly, to **apply** each theorem appropriately in problems. The latter means: decide which theorem to use, check (in writing!) that all conditions of your theorem are satisfied in the problem in question, and then state (in writing!) the conclusion of the theorem using the specifics of your problem.

(1) **The Fundamental Theorem of Calculus:** The definite integral of $f(x)$ equals any antiderivative $F(x)$ of $f(x)$, evaluated at the ends of the interval, i.e. $\int_a^b f(x)dx = F(b) - F(a)$ where $F'(x) = f(x)$.
(2) **Table for Direct Integration:** must be committed to memory for efficient direct integration. Summarize your classnotes and the textbook references in a table which includes the most common

directly integrable functions. Do **not** miss formulas 17-18 in §7.5: you do not want to rederive these formulas on the exam, but rather quickly use them.

(3) **Substitution in Indefinite Integrals:** $\int f(g(x))g'(x)dx = \int f(u)du$ $(u = g(x),\ du = u'(x)dx.)$

(4) **Substitution in Definite Integrals:** $\int_a^b f(g(x))g'(x)dx = \int_{u(a)}^{u(b)} f(u)du$ $(u = g(x),\ du = u'(x)dx.)$

(5) **Integration by Parts for indefinite integrals:** $\int udv = uv - \int vdu$, or:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx.$$

(6) **Integration by Parts for definite integrals:** $\int_a^b udv = uv\big|_a^b - \int_a^b vdu$, or

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\big|_a^b - \int_a^b g(x)f'(x)dx.$$

(7) **Trigonometric Identities.**
   (a) Pythagorus Theorem: $\sin^2 x + \cos^2 x = 1$; replacement: $\cos^2 x = 1 - \sin^2 x;\ \sin^2 x = 1 - \cos^2 x$.
   (b) Half-angle formulas (reduction of degree): $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$.
   (c) Double-angle formulas: $\sin 2x = 2\sin x\cos x,\ \cos 2x = \cos^2 x - \sin^2 x$.
   (d) Turning products into sums: $\sin x\cos y = \frac{1}{2}(\sin(x-y) + \sin(x+y))$;
   $\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$; $\cos x\cos y = \frac{1}{2}(\cos(x-y) + \cos(x+y))$.

(8) **Partial Fraction Representation Theorem.** Any rational function $f(x)/g(x)$ can be written uniquely as a sum of simpler fractions. Specifically, any linear factor of $g(x)$ contributes as follows:

$$\frac{f(x)}{(x-a)^n} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{A_3}{(x-a)^3} + \cdots + \frac{A_{n-1}}{(x-a)^{n-1}} + \frac{A_n}{(x-a)^n}.$$

And any irredicible quadratic factor contributes as follows:

$$\frac{f(x)}{(ax^2+bx+c)^n} = \frac{B_1x + C_1}{ax^2+bx+c} + \frac{B_2x + C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_{n-1}x + C_{n-1}}{(ax^2+bx+c)^{n-1}} + \frac{B_nx + C_n}{(ax^2+bx+c)^n}.$$

However, if $\deg f(x) \geq \deg g(x)$, then *long division* is in order to peel off a polynomial *before* any of the above partial fraction representations are used. To actually integrate one of the partial fractions with an irreducible quadratic in denominator, one must complete the square in the denominator, make a mild linear substitution, and then split the integral appropriately along the numerator before finally integrating.

(9) **Trapezoidal and Simpson's Approximations.** We can approximate the area under a continuous function $f(x)$ on $[a,b]$ by trapezoids and by parabolas:
   (a) $T_n = [f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)] \cdot \frac{\Delta x}{2}$.
   (b) $S_{2n} = [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(x_{2n})]\ \frac{\Delta x}{3}$.
   Note that Simpson's approximation works only with even number of subintervals of $[a,b]$, hence the even notation $2n$ for the index of S.

(10) **Error Bounds for Integral Approximations.** Let $f(x)$ be a function on $[a,b]$ whose second derivative can be bounded by a number K: $|f''(x)| \leq K$ for all $x \in [a,b]$. Then the errors E_{T_n} and E_{M_n} of the trapezoidal and midpoint approximations of $\int_a^b f(x)dx$ are bounded as follows:

$$|E_{T_n}| \leq \frac{K(b-a)^3}{12n^2},\ \text{and}\ |E_{M_n}| \leq \frac{K(b-a)^3}{24n^2}.$$

If the fourth derivative of $f(x)$ can be bounded by a number N: $|f^{(4)}(x)| \leq N$ for all $x \in [a,b]$, then the error E_{S_n} of the Simpson's approximation of $\int_a^b f(x)dx$ is bounded as follows:

$$|E_{S_n}| \leq \frac{N(b-a)^5}{180n^4}\ \text{(here } n \text{ can be only even.)}$$

Note that the above formulas are **not** approximations of the integral, neither are they the exact errors of T_n, M_n or S_n, respectively. These formulas give us an **upper bound** for the errors in these approximations, i.e. they tell us the **maximum (worse) error** that each of T_n, M_n or S_n

can have. The **actual** errors could be much smaller, but usually there is no way to find the actual errors; hence, we contend ourselves with knowing a bound for the errors.

(11) **Comparison Theorem for improper integrals.** If $f(x)$ and $g(x)$ are *continuous* functions on $[a,\infty)$, $(-\infty,b]$ or $(-\infty,\infty)$, and $f(x)\ge g(x)\ge 0$ for *all* x in the domain interval, and
  (a) If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ also converges, and $\int_a^\infty g(x)\,dx < \int_a^\infty f(x)\,dx$.
  (b) If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ also diverges.

Note that the above is written only for the interval $[a,\infty)$, but analogous statements work for any infinite inverval.

(12) **Arc Length Formula** and **Surface of Revolution.** If $f(x)$ is a function on $[a,b]$ such that $f'(x)$ is continuous on $[a,b]$, then the *arc length* L of the curve $y=f(x)$ is given by

$$L=\int_a^b\sqrt{1+(f'(x))^2}\,dx;$$

and the *surface area* S of the solid obtained by revolving $f(x)$ about the x-axis is given by

$$S=\int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx.$$

If $f'(x)$ is discontinuous or undefined somewhere on $[a,b]$ (including the ends a or b), the above integrals should be treated as indefinite, and the standard modifications of the integrals via limits must be applied. Be aware of the change of formulas if the function is given in terms of y: $x=g(y)$, and/or the function is revolved about the y-axis.

(13) **Probabilities and Areas.** If $f(x)$ is the probability density function of a continuous random variable X, the probability that X will attain values on the interval $[a,b]$ is given by

$$P(a\le X\le b)=\int_a^b f(x)dx.$$

In particular, $P(-\infty\le X\le+\infty)=\int_{-\infty}^\infty f(x)dx=1$.

(14) **Exponential Density Function.** Every exponential density function has the form $f(t)=ce^{-ct}$ when $t\ge 0$, and $f(t)=0$ for $t<0$. Its mean is $\mu=1/c$, so that $f(t)$ can be rewritten as

$$f(t)=\begin{cases}0 & \text{if } t<0\\ \frac{e^{-t/\mu}}{\mu} & \text{if } t\ge 0.\end{cases}$$

(15) **Mean, Median and Standard Deviation** of a probability density function $f(t)$ of a continuous random variable X are computed as follows:
  (a) the mean $\mu=\int_{-\infty}^\infty xf(x)\,dx$;
  (b) the median is the number m for which $\int_m^\infty f(x)\,dx=\frac{1}{2}$.
  (c) the standard deviation $\sigma=\sqrt{\int_{-\infty}^\infty (x-\mu)^2 f(x)\,dx}$, where μ is the mean of X.

(16) **(bonus) Normal Distribution** corresponds to a continuous random variable X with a density function

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-(x-\mu)^2/(2\sigma^2)}.$$

Here the constants μ and σ are equal to the mean and standard deviation of X. (Try to check this out by using the definitions of mean and standard deviation.) Due to the fact that $f(x)$ is symmetric with respect to the vertical line $x=\mu$, the median m coincides with the mean μ.

3. Problem Solving Techniques

(1) **How do we guess $u(x)$ and $u'(x)$ in the substitution rule?** There isn't any "bullet-proof" method for guessing (that's why it's called "guessing" afterall), but here are several tips that will suffice in all problems of Calculus II. By applying the Substitution Rule we are making an attempt to reverse the effect of differentiation by the Chain Rule. Thus, we want to represent our function as

if it were the result of a Chain Rule: $f(g(x)) \cdot g'(x)$, where $u = g(x) =$ blah; in other words, we are trying to write our function as a product of a "u-part" and a "$u'(x)dx$-part". The key to "correct guessing" is to remember that, whatever $u(x)$ turns out to be, the derivative $u'(x)$ **must appear multiplied by** dx!

(a) Try to locate $u'(x)$: all candidates for $u'(x)$ are among the things with which dx is multiplied.

(b) For example, $\int 3x^2 \cos(x^3)dx$ yields two possibilities for $u'(x)$: $u'(x) = 3x^2$ and $u'(x) = \cos(x^3)$. Now let's recall that we need to determine also $u(x)$, i.e. we have to do direct integration on our chosen $u'(x)$. Obviously, the choice $u'(x) = \cos(x^3)$ is bad because we don't know how to directly integrate $\cos(x^3)$. However, the choice $u'(x) = 3x^2$ works very well, since we can integrate immediately: $u(x) = x^3 + C$, and notice that x^3 appears in the rest of the expression. Thus, we set $u(x) = x^3$ and $du = (x^3)'dx = 3x^2dx$, and the substitution rule yields:

$$\int 3x^2 \cos(x^3)dx = \int \cos(x^3)(3x^2dx) = \int \cos(u)du = \sin(u) + C = \sin(x^3) + C.$$

Don't forget to check the answer by differentiation!!

(c) Recall that "denominators" may also yield something relevant for $u'(x)$. For example, in $\int \frac{\arctan x}{1+x^2}dx$ it will be unsuitable to set $u'(x) = \arctan(x)$ (why?), but it will be very suitable to set $u'(x) = \frac{1}{1+x^2}$: then $u(x) = \int \frac{1}{1+x^2}dx = \arctan(x)$ and $du = \frac{1}{1+x^2}dx$ so that

$$\int \frac{\arctan x}{1+x^2}dx = \int \arctan x \cdot \left(\frac{1}{1+x^2}dx\right) = \int udu = \frac{u^2}{2} + C = \frac{\arctan^2(x)}{2} + C.$$

Don't forget to check the answer by differentiation!!

(d) Sometimes we have readjust $u'(x)$ and $u(x)$ by constants. For example, in $\int x^2 \cos(x^3 - 7)dx$ we can initially guess $u'(x) = x^2$ so that $u(x) = \frac{x^3}{3} + C$. But this $u(x)$ does **not** conveniently appear in the rest of the function! Instead, we would have liked that $u(x) = x^3 - 7$: this is a harmless little wish, which can be satisfied on the spot by setting $du = (x^3 - 7)'dx = 3x^2dx$. We can then "solve" $du = 3x^2dx$: $x^2dx = \frac{1}{3}du$, and substitute in the integral:

$$\int x^2 \cos(x^3 - 7)dx = \int \frac{1}{3}\cos(u)du = \frac{1}{3}\sin(u) + C = \frac{1}{3}\sin(x^3 - 7) + C.$$

Don't forget to check the answer by differentiation!!

(e) We learn from the above that after having made a choice for $u'(x)$, and hence finding $u(x)$, we can readjust $u(x)$ by replacing it by any convenient expression of the form $C_1u(x) + C_2$ where C_1 and C_2 are some suitable constants. This forces a slight change in $u'(x)$: multiplication by the constant C_1. To summarize: always keep in mind that certain degrees of freedom in our choices for $u(x)$ and $u'(x)$ exist: for $u(x)$ there are "2 degrees of freedom", and for $u'(x)$ there is only "1 degree of freedom". Be aware that among all possible choices for $u(x)$ and $u'(x)$ there is one most convenient choice: it is dictated by what we would like $u(x)$ to be in our original function.

(f) Sometimes it may appear as if there are **no** candidates for the role of $u'(x)$. For example, in $\int e^{6x}dx$, dx is multiplied only by e^{6x}, so what can $u'(x)$ be? It can be a constant: it can be any constant that we like. So, we look for a possible $u(x)$: it looks like $u = 6x$ would be nice; hence $u'(x) = 6dx$, $dx = \frac{1}{6}du$, and

$$\int e^{6x}dx = \int \frac{1}{6}e^u du = \frac{1}{6}e^u + C = \frac{1}{6}e^{6x} + C.$$

Don't forget to check the answer by differentiation!!

(g) Finally, some problems are trickier than we would have liked. The "trouble" there, surprisingly, is not in the guessing of $u(x)$ and $u'(x)$, but rather in trying to get rid of all x's and replacing them by u's. For example, in $\int \sqrt{x^2 + 1}x^5dx$, it is clear that we definitely want to get rid of $x^2 + 1$ under the radical, so we substitute $u(x) = x^2 + 1$, $du = 2xdx$, but then how do we get rid of the remaining x^4 in our function? We solve for x^4 from our substitution equation: $u = x^2+1 \Rightarrow x^2 = u-1 \Rightarrow x^4 = (u-1)^2$, so that our integral will look like:

$$\int \sqrt{x^2+1}x^5dx=\int \sqrt{x^2+1}x^4xdx=\int \sqrt{u}(u-1)^2\cdot\frac{du}{2}=\int \frac{1}{2}\sqrt{u}(u^2-2u+1)du=\frac{1}{2}\int(u^{5/2}-2u^{3/2}+u^{1/2})du$$

Finish this example as we did in class, and check with differentiation.

(h) Continuing with the "mean streak" of substitution rule problems, check out some of the review problem in Chapter 5. Don't forget also the trick in #41 (§5.5) of splitting the integral as a sum of two integrals; also, don't forget that functions maybe even/odd, thus simplifying integration on symmetric intervals $[-a, a]$.

(2) **How do we locate u and v in Integration by Parts?** The idea of Integration by Parts is to represent our function as a product of two functions $f(x)$ and $g'(x)$ such that it is **easier** to integrate $\int f'(x)g(x)dx$ than the original integral $\int f(x)g'(x)dx$. It is usually obvious how to represent our original function as a product of two functions. What is **not** obvious is to decide which of these functions will play the role of $u = f(x)$ and which will play the role of $v = g(x)$. As a rule of thumb, the function which has "simpler" derivative plays the role of $u = f(x)$, and the function which has simple enough integral plays the role of $v = g(x)$.

(a) Polynomials are excellent candidates for $u = f(x)$ because their derivatives are *simpler* functions: they are polynomials of lower degrees!

(b) The classical inverse functions such as inverse trig functions ($\arcsin(x)$, $\arccos(x)$, $\arctan(x)$, $\text{arccot}(x)$) and logarithmic functions ($\ln(x)$) are also excellent candidates for $u = f(x)$ because we know nothing about their integrals, but we do know a lot about their derivatives.

(c) Exponential functions (e.g. e^x) and trigonometric functions like $\sin x$ and $\cos x$ can be used both as $u = f(x)$ and $v = g(x)$, depending on the "needs" of the other involved function. The reason for this "neutral" behavior of e^x, $\sin x$ and $\cos x$ is that both their derivatives and integrals are as "complicated" as the original functions themselves.

(d) Example of a product of a polynomial and a trig. function: $\int x^3 \cos x dx$. We set $u = x^3$ because the derivative $u' = 3x^2$ is an easier function. Hence $v' = \cos x$, $v = \sin x$.

$$\int x^3 \cos x dx = uv - \int v du = x^3 \sin x - \int 3x^2 \sin x dx.$$

We can now again apply Integration by Parts on the last integral by setting $u = 3x^2$, $v' = \sin x$. This will further reduce the degree of the polynomial: we'll end up having to find $\int 6x \cos x dx$. One last integration by parts with $u = 6x$ and $v' = \cos x$ will reduce everything to finding $\int 6 \sin x dx$. Make sure you complete this whole calculation carefully and substitute back all your answers into the original integral.

(e) Example of a product of a polynomial and an exponential function: $\int xe^x dx$. By analogy with (d), we set $u = x$ and $v' = e^x$, so that $u' = 1$ and $v = e^x$:

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C.$$

The idea here was again to reduce the power of the polynomial.

(f) What if there is no product? For example, $\int \arcsin x dx$? We recall one of the greatest tricks in mathematics: multiplying by 1! (The other trick is "adding 0". :)) Thus, we can set $u = \arcsin x$, $v' = 1$, so that $u' = 1/\sqrt{1-x^2}$ and $v = x$:

$$\int \arcsin x dx = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx.$$

Now we use the substitution rule: set $u = 1 - x^2$ so that $du = -2xdx$, and

$$\int \frac{x}{\sqrt{1-x^2}} dx = \int -\frac{1}{2\sqrt{u}} du = \int -\frac{1}{2} u^{-1/2} du = -u^{1/2} + C = -\sqrt{1-x^2} + C.$$

Finally,

$$\int \arcsin x dx = x \arcsin x + \sqrt{1-x^2} + C.$$

Check by differentiation!!

(g) An example of a product of an exponential and a trigonometric function: $\int e^x \cos x dx$. Since neither e^x nor $\cos x$ has a simpler derivative (or integral), it doesn't matter how we apply integration by parts here. So, say, $u = e^x$, $v' = \cos x$, $u' = e^x$, $v = \sin x$:

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx.$$

OK, we didn't get anything simpler, but what if we venture once again and do a similar integration by parts: $u = e^x$, $v' = \sin x$, so that $u' = e^x$, $v = -\cos x$:

$$\int e^x \sin x dx = -e^x \cos x + \int e^x \cos x dx.$$

Substituting above,

$$\int e^x \cos x dx = e^x \sin x - (-e^x \cos x + \int e^x \cos x dx) = e^x(\sin x + \cos x) - \int e^x \cos x dx$$

$$\Rightarrow \int e^x \cos x dx = \frac{1}{2} e^x (\sin x + \cos x).$$

(h) For extra challenge, review Ex. 6 in §7.1, and do #42-44 in §7.1.

(3) **When do we decide to use a trigonometric substitution?** When the integrand has a radical of the form: $\sqrt{x^2+a^2}$, $\sqrt{a^2-x^2}$ or $\sqrt{x^2-a^2}$. Each of these radicals requires a different trig. substitution. Check your classnotes for details on the individual substitutions. It is always useful to draw a right triangle and label it appropriately according to the Pythagorus theorem, so that the unwanted radical is expressed as one of the sides of the triangle. The goal of the trigonometric substitution is to end with a trig. integral, e.g. for some integer n and m:

$$\int \sin^n x \cos^m x \, dx. \tag{1}$$

A word of caution: sometimes, radicals can be dealt with by a direct u-substitution, even if they contain radicals of the above form. So, before you embark on a trig. substitution, make sure you are not missing a simpler substitution and a faster way to solve the problem.

(4) **How do we compute trigonometric integrals?** By "trig. integrals", we understand integrals of form in (1). We have studied two methods for solving trig. integrals:

(a) *Peel off, replace and do u-substitution:* this works if one (or both) trig. functions appear in odd powers. For example, if the sine function appears in odd power (e.g. $\sin^7 x$), then peel off one sine as $\sin x \, dx$, replace the remaining sines (e.g. $\sin^6 x = (\sin^2 x)^3 = (1-\cos^2 x)^3$), and substitute the cosine function (e.g. $u = \cos x$, $du = -\sin x \, dx$):

$$\int \sin^7 x \cdot \cos^4 x \, dx = \int \sin^6 x \cdot \cos^4 x \sin x \, dx = \int (1-\cos^2 x)^3 \cos^4 x \sin x \, dx = -\int (1-u^2)^3 u^4 \, du.$$

Multiply through and integrate the resulting polynomial directly. Don't forget to substitute back $\cos x$. As a general rule, whatever function you peel off, you'll end up substituting the **other** function.

If you have **both** sine and cosine appearing in odd powers, usually it is better to peel off the function in the smaller power in order to have a simpler multiplication of polynomials at the end:

$$\int \sin^7 x \cdot \cos^5 x \, dx = \int \sin^7 x \cdot \cos^4 x \cos x \, dx \text{ (finish it off! don't be lazy!)}$$

(b) *Reduce powers, multiply through,* then see if peel off will work or again reduction of power, and so on. Reduction of powers works only if **both** trig. functions appear in even powers, e.g.

$$\int \sin^6 x \cdot \cos^4 x\, dx = \int (\sin^2 x)^3 \cdot (\cos^2 x)^2 dx = \int \left(\frac{1-\cos(2x)}{2}\right)^3 \left(\frac{1+\cos(2x)}{2}\right)^2 dx =$$

$$= \frac{1}{2^5}\int \big(1 - 3\cos(2x) + 3\cos^2(2x) - \cos^3(2x)\big)\big(1 + 2\cos(2x) + \cos^2(2x)\big) dx =$$

$$= \frac{1}{2^5}\int \big(1 - \cos(2x) - 2\cos^2(2x) + 5\cos^3(2x) - 2\cos^4(2x) - \cos^5(2x)\big) dx =$$

$$= \frac{1}{2^5}\int \big(-\cos(2x) + 5\cos^3(2x) - \cos^5(2x)\big) dx + \int \big(1 - 2\cos^2(2x) - 2\cos^4(2x)\big) dx$$

The integrals in the first batch will be dealt with by "peel off", while the integrals in the second batch will fall under "reduction of power". (Why don't you check the multiplications above and let me know if there is a typo?... Checking who's reading this... :))

Note that if the integrand has only one trig. function, it still falls within one of the two methods described above, e.g. $\int \sin^5 x\, dx$ will "peel off", while $\int \sin^6 x\, dx$ will require reduction of the power 6.

A further finessing of the above method can be done if the integrand is a **fraction** of trig. functions. If an odd power is in the numerator, the "peel off" method will turn the integrand into a rational function, and then splitting of the integral will finish the job, e.g.

$$\int \frac{\cos^3 x}{\sin^4 x} dx = \int \frac{\cos^2 x}{\sin^4 x} \cos\, x dx = \int \frac{1-\sin^2 x}{\sin^4 x} \cos\, x dx = \int \frac{1-u^2}{u^4} du = \int u^{-4} du - \int u^{-2} du...$$

If an odd power is in the denominator, then we can force a "peel off" in the numerator by multiplying appropriately top and bottom:

$$\int \frac{\cos^4 x}{\sin^3 x} dx = \int \frac{\cos^4 x}{\sin^4 x} \sin\, x dx = \int \frac{\cos^4 x}{(1-\cos^2 x)^2} \sin\, x dx = -\int \frac{u^4}{(1-u^2)^2} du = -\int \frac{u^4}{(1-u)^2(1+u)^2} du.$$

The method of partial fractions will finish the job here.

(5) **How do we apply partial fractions to integration?** The method of partial fractions applies when the integrand is a rational functions, i.e. a ratio of two polynomials as in $\displaystyle\int \frac{f(x)}{g(x)} dx$.

(a) If $\deg f(x) \geq \deg g(x)$, then apply long division to peel off a polynomial: $\dfrac{f(x)}{g(x)} = h(x) + \dfrac{r(x)}{g(x)}$. Here $h(x)$ is the *result* of the division of $f(x)$ by $g(x)$, and $r(x)$ is the *remainder* of this division. Thus, the integral becomes:

$$\int \frac{f(x)}{g(x)} dx = \int \left(h(x) + \frac{r(x)}{g(x)}\right) dx = \int h(x) dx + \int \frac{r(x)}{g(x)} dx.$$

We know how to integrate $h(x)$, while $\deg r(x) < \deg g(x)$, so the second integral is solved using partial fractions as described below.

(b) If $\deg f(x) < \deg g(x)$, factor $g(x)$ as a product of linear and irreducible quadratic polynomials. Then use the Partial Fraction Representation Theorem to write the integrand $f(x)/g(x)$ as a sum of several simple fractions. Next, get rid of all denominators, equate all coefficients and make a system of linear equations. Solve this system to find the unknown constants A, B, C,... Finally, plug these constants into your partial fraction representation of $f(x)$ and integrate each such fraction separately (see (c) below).

Warning: say, you are integrating $\int \frac{x^4-2x^3+7}{(x-1)^2(x+2)} dx$. A JEOPARDY question: The degree of the top is bigger than degree of the bottom, so long division is in order; what should we do first: the long division and then partial fractions, or first partial fractions and then long division? Answer: neither of the above because, even though we have to do long division before partial fractions, we **cannot** long divide by a "factored" polynomial. So, multiply through the denominator, then do long division, and then the new fraction will have the same denominator, so you can factor it back as it was before:

$$\int \frac{x^4-2x^3+7}{(x-1)^2(x+2)}dx = \int \frac{x^4-2x^3+7}{x^3-3x+2}dx = \int (x+1)dx + \int \frac{x+5}{x^3-3x+2}dx =$$

$$= \frac{x^2}{2} + x + \int \frac{x+5}{(x-1)^2(x+2)}dx.$$

Only now the last fraction is in a form suitable for partial fractions. Is there a mistake above? Finish the integration.

(c) Recall the direct and u-substitution integration formulas:

$$\int \frac{1}{(x-a)^n}dx = \frac{(x-a)^{-n+1}}{-n+1} + C \text{ when } n \neq 1; \quad \int \frac{1}{(x-a)}dx = \ln(x-a) + C;$$

$$\int \frac{1}{(x^2+a^2)}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C; \quad \int \frac{x}{(x^2+a^2)^n}dx = \frac{1}{2}\int \frac{du}{u^n} = ...$$

What was substituted in the last integral? Finish the integration there and make it into a formula to use for partial fractions in case of dire need. :)

(d) Sometimes irreducible quadratic factors require *completion of the square*, as in $\int \frac{x+2}{3x^2-12x+24}dx$:

$$3x^2 - 12x + 24 = 3(x^2 - 4x + 8) = 3(x^2 - 4x + 4 + 4) = 3((x-2)^2 + 4).$$

So, our integral becomes:

$$\int \frac{x+2}{3((x-2)^2+4)}dx = \frac{1}{3}\int \frac{u+4}{u^2+4}du = \frac{1}{3}\int \frac{u}{u^2+4}du + \frac{1}{3}\int \frac{4}{u^2+4}du.$$

where $u = x-2$, $du = dx$, $x = u+2$, and we used splitting of the fraction for the last equality. Now, for the first integral we will use substitution: $t = u^2+4$, while for the second integral we will use a direct formula of integration (involving arctan). Finish the integration and don't forget to substitute back.

The general formula for completion of a square is as follows:

$$x^2 + bx + c = x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c = \left(x + \frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2.$$

In other words, what you "complete the square with", $\left(\frac{b}{2}\right)^2$, is obtained by taking the coefficient b of x, dividing it by 2, and then squaring the fraction $b/2$. **Warning**: the above formula works only if the coefficient of x^2 is 1! If x^2 has some other coefficient, e.g. $3x^2$ as in the example above, then you must first factor out 3 from the whole expression, and then apply the formula for completion of the square.

(e) The careful reader will have noticed that one type of fraction is missing from the formulas in (c) (this is just for bonus):

$$\int \frac{1}{(x^2+a^2)^n}dx = \frac{A_1}{x^2+a^2} + \frac{A_2}{(x^2+a^2)^2} + \frac{A_3}{(x^2+a^2)^3} + \cdots + \frac{A_{n-1}}{(x^2+a^2)^{n-1}} + A_n\arctan(x^2+a^2) + C.$$

This is of no use unless we find out the constants $A_1, A_2, ..., A_n$. To this end, differentiate RHS and set this derivative to equal the integrand $1/(x^2+a^2)$. Then follow the well-known path: get rid of denominators, equate coefficients to obtain a system of equations, solve this system for $A_1, A_2, ..., A_n$, and voila - plugging these numbers above yields the integral we were looking for.

(6) **How do we decide which method to use in solving integrals?** This is by far the hardest, but most rewarding, part of solving integrals. You are given a problem, but not given a strategy or hint how to do it. It is up to you to make educated choices and try them out until one works out. Think "ahead of the game": what kind of functions **can** you integrate for sure, so try to reduce the given integrands to functions that you like.

(a) *Simplify the integrand if possible.* This may involve one or more of the following: cancelling, multiplyling through, replacing trig. functions only by the basic trig. functions $\sin x$ and $\cos x$, replacing expressions using formulas (e.g. double angle formulas), etc.

(b) *Look for obvious substitutions*: this means that you see some function in the **numerator**, which you identify as the derivative of your substitution, e.g.

$$\int \frac{\sin x}{2\cos x + 7} dx = \int \frac{-1}{2u} du.$$

What did I substitute? Finish the problem.

(c) *Describe the type of function you are integrating and try to find a method that corresponds to this type of functions.* (This is only if the first two steps didn't yield a solution.)

(i) *Trigonometric functions*: $\sin^n x \cos^m x$ for integers n and m. Use the methods described earlier: peel off or reduction of power.

(ii) *Rational functions*: $f(x)/g(x)$ for two polynomials $f(x)$ and $g(x)$. Use partial fractions (with long division, if necessary).

(iii) *Product of functions*: $f(x) \cdot g(x)$. Try integration by parts: you need to decide which will be u and which will be v. As a general rule, u should be the function whose **derivative** is a simpler function, while v should be a function whose **antiderivative** is not hard to find even though it may be more complicated than v itself. Look at your classnotes for examples of different types in which we decided on u and v.

Even if there is only one function, $f(x)$, we may sometimes find it advantageous to think of it as a product of $1 \cdot f(x)$, and set $u = f(x)$ and $v = 1$. This is especially helpful with "horrible" functions like $f(x) = \ln x$, $\arctan x$, $\arcsin x$, etc. We would really like to get rid of these functions and replace them by their derivatives, and integration by parts will do exactly that.

(iv) *Radicals.* These can be dealt with in different ways.

(A) $\sqrt{x^2 + a^2}$, $\sqrt{a^2 - x^2}$ or $\sqrt{x^2 - a^2}$ usually call for trig. substitution.

(B) $\sqrt[n]{f(x)}$ usually calls for a "full" rationalizing substitution: i.e. $u = \sqrt[n]{f(x)}$, $u^n = f(x)$, $n\, u^{n-1} du = f'(x)dx$. Note how we dealt with the differential operators du and dx: this is a shortcut, which proves to be very useful in such problems and saves a lot of trouble.

(C) $\sqrt[n]{f(x)}$ sometimes calls for an "under-the-radical" substitution: i.e. $u = f(x)$, $du = f'(x)dx$, etc.

(D) Rationalizing numerator or denominator also counts as a way to deal with radicals. Recall how we rationalize expressions of the form $\sqrt{\ldots} \pm \sqrt{\ldots}$.

(d) If everything else fails,

(i) *Try unobvious substitution:* this means that you don't yet see $u'(x)$ in the numerator, but you still would like to substitute some function $u = f(x)$. Try it out and deal with the consequences later. It may or may not work out. A prime example of this is Ex.2 in §7.5.

(ii) *Do something unexpected:* manipulate (legally!) the integrand, e.g. force a "rationalizing" substitution of $(1 + \sin x)$ in the numerator by multiplying top and bottom by $(1 - \sin x)$; use trig. identities, *split* the integral into two or more integrals using splitting of fractions, etc.

(iii) *Recall an earlier problem* to which you can reduce the present integral.

(iv) *Don't give up.* Try all over again from the beginning. You must have missed something since the exam problem can't be *that* hard!

(7) **How do we find the bounds K and N** for $f''(x)$ and $f^{(4)}(x)$ on $[a, b]$ in order to apply the error bound formulas for T_n, M_n and S_n? There is no single recipe for finding K and N, just like there is no single way for proving inequalities. Here are some ideas, which by no means summarize all possibilities.

(a) If $f(x)$ is a fraction $f(x) = g(x)/h(x)$, try to find an upper bound for the numerator and a lower bound for the denominator, i.e. $|g(x)| \leq A$ and $|h(x)| \geq B$ for **all** $x \in [a, b]$. Then conclude that $|f(x)| \leq A/B$. In other words, "increasing the numerator and decreasing the denominator" will increase the fraction. Warning: we have used the absolutely value signs everywhere on the involved functions in order to make sure that all involved numbers are positive and hence there is no change of inequality signs on the way to obtaining the bounds A, B and A/B.

(b) When $\sin x$ and $\cos x$ are involved, the well-known bounds $|\sin x| \leq 1$ and $|\cos x| \leq 1$ for all x are sometimes useful.

(c) When $f(x)$ is increasing on $[a, b]$ (e.g. check $f'(x) \geq 0$ for all $x \in [a, b]$), then $f(a) \leq f(x) \leq f(b)$, i.e. minimum occurs in the beginning and maximum occurs at the end of the interval. This observation applies equally well to parts of $f(x)$ which are obviously increasing. (Analogous ideas apply if the involved functions are decreasing.: the maximum is obtained at the beginning of the interval.) For instance, $f(x) = (x^2 + 5)/(x - 2)$ on $[-4, -3]$ can be bounded as follows: since $x^2 + 5$ is positive and decreasing on $[-4, -3]$ (why?) and $x - 2$ is negative and increasing on $[-4, -3]$ (why?), then $x^2 + 5$ attains its maximum at the beginning $x = -4$ of the interval, and $|x-2|$ being positive and decreasing, attains its minimum at the end $x = -3$ of the interval, i.e. $|x^2 + 5| = x^2 + 5 \leq (-4)^2 + 5 = 21$ and $|x - 2| \geq |-3 - 2| = 5$. Using again the idea of "increasing the numerator and decreasing the denominator", we arrived at

$$|f(x)| = \frac{|x^2 + 5|}{|x - 2|} \leq \frac{21}{5} \text{ for all } x \in [-4, -3].$$

(d) When $f(x)$ or part of it involves sums/differences of other functions, then the *triangle inequality* is useful: $|a \pm b| \leq |a| + |b|$ and $|a \pm b| \geq |a| - |b|$. For instance, if $f(x) = (\sin x - 7x)/(x^3 - 4x + \cos x)$ on $[10, 20]$,

$$|f(x)| = \frac{|\sin x - 7x|}{|x^3 - 4x + \cos x|} \leq \frac{|\sin x| + |7x|}{|x^3 - 4x + \cos x|} \leq \frac{1 + 7 \cdot 20}{10^3 - 4 \cdot 10 - 1} = \frac{141}{959}.$$

Note that in the denominator we used that $x^3 - 4x$ is increasing on $[10, 20]$ (why? check derivative), hence its minimum value is obtained at $x = 10$, while the minimum of $\cos x$ is -1; so putting this together, we arrived at $x^3 - 4x + \cos x \geq (10^3 - 4 \cdot 10) - 1 = 959$.

(8) **How do we use the error bounds for T_n, M_n and S_n?** If a problem asks to calculate $\int_a^b f(x)dx$ accurate to within 0.01 using, say, Simpson's approximations, we must go through several steps:

(a) Calculate $f^{(4)}(x)$;

(b) Find an upper bound N for $|f^{(4)}(x)|$ on $[a, b]$, i.e. $|f^{(4)}(x)| \leq N$ for all $x \in [a, b]$;

(c) Set the error bound inequalities

$$|E_{S_n}| \leq \frac{N(b - a)^5}{180n^4} \leq 0.01$$

and solve for n:

$$\sqrt[4]{\frac{N(b - a)^5}{180 \cdot 0.01}} \leq n.$$

Round up the number on the LHS (left-hand side) to the nearest integer, e.g. if $892.37 \leq n$, we round up to $n = 893$. Yet, for the Simpson's approximation we need n to be even, so $n = 894$.

(d) Calculate S_n for your answer for n from (c), e.g. S_{894}. Conclude that S_{894} is within 0.01 of the exact value of $\int_a^b f(x)dx$.

Devise similar algorithms when using midpoint M_n or trapezoidal T_n approximations.

(9) **How do we compute improper integrals? How do we decide when they converge or diverge?**

(a) First, decide if indeed your integral is improper. There could be one or more of the following reasons.

(i) The interval of integration is infinite, e.g. $[a, \infty)$, $(-\infty, b]$, or $(-\infty, +\infty)$. In each case, we turn the integral into a limit ordinary (proper) integrals by replacing the $\pm\infty$ by a new variable t, and letting $t \to \pm\infty$:

$$\int_a^{\infty} f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx; \quad \int_{-\infty}^{b} f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx;$$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_c^{\infty} f(x)dx = \lim_{t\to-\infty} \int_t^c f(x)dx + \lim_{t\to\infty} \int_c^t f(x)dx.$$

Note that the splitting point c in the third case can be chosen as it is convenient for our calculations, so it is our responsibility to choose a suitable c. If the function $f(x)$ is odd or even, it is reasonable to choose $c = 0$.

(ii) The function $f(x)$ has an infinite discontinuity in the interval of integration $[a, b]$. So, find where $f(x)$ has this discontinuity, say, at $x = c$, split the integral into two parts, replace c in each part by t, and let $t \to c^+$ and $t \to c^-$, as appropriate:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx = \lim_{t\to c^-} \int_a^t f(x)dx + \lim_{t\to c^+} \int_t^b f(x)dx.$$

What one has to remember here is again that we represent the original improper integral as a limit of ordinary proper integrals.

(b) After you have determined why the integral is improper and you have replaced it by limits of proper integrals, it is obvious how to proceed: calculate each (proper) integral using standard integration techniques, and then take the corresponding limits to figure out the final value of the original improper integral.

(c) In case there are two or more limits of proper integrals involved, do not forget about the **"iron curtain"** between the various integrals that come up in the calculation: you have to perform all calculations separately on both sides of the "curtain". If you get even one infinity (plus or minus) somewhere, then forget the whole thing - the original integral **diverges** and that's it!

The only case when the "iron curtain" will go down, is when everywhere you get finite numbers: only then you are allowed to put them together to come up with the final value for the original improper integral (which will be **convergent** in this case.)

(10) **How do we use the Comparison Theorem for improper integrals?** Say, you are trying to find whether $\int_{-\infty}^{b} f(x)dx$ converges or diverges, but you cannot (or do not want to) compute the antiderivative of $f(x)$. The only solution is if you know of **another** improper integral which can be compared to the original integral. Thus, look carefully at your function $f(x)$: can you simplify it to some similar function $g(x)$ whose antiderivative (and hence integral) you know how to find easily? Examples of such "twin" functions are as follows: $f(x) = e^{-x^2}$ and $g(x) = e^{-x}$, $f(x) = x^{4.78}$ and $g(x) = x^5$ or $g(x) = x^4$, etc.

In any case, calculate $\int_{-\infty}^{b} g(x)dx$, and then compare **both** $g(x)$ to $f(x)$, and $\int_{-\infty}^{b} g(x)dx$ to $\int_{-\infty}^{b} f(x)dx$. Be careful when the Comparison Theorem applies and when it doesn't: e.g. can we conclude anything for sure about $\int_a^{\infty} f(x)\,dx$ if $\int_a^{\infty} g(x)\,dx$ diverges **and** $0 \le f(x) \le g(x)$? if $\int_a^{\infty} g(x)\,dx$ converges **and** $0 \le g(x) \le f(x)$?

(11) **How do we compute arc lengths and surfaces of revolution?** We use the Arc Length Formula and the Surface of Revolution Formula, respectively. To set up the integral, first decide wrt which variable you will be integrating. Usually, if the function is given in variable x, e.g. $y = x^3$ or $f(x) = x^3$, this suggests the variable x. But be careful with surfaces of revolution: if the problems states that the body is obtained via revolution about, say, the y-axis, we have no choice but to integrate wrt y. If given a *curve* equation, we need to solve it for x or y to get a function; sometimes this results in two or more functions (e.g. in the case of a circle or ellipse), so we must decide for

which branch of the curve we want to find the arc length/surface of revolution. (Compare with your classnotes.)

Do not forget to use the **derivative** $f'(x)$ in the arc length formula. The integrand that results will be a radical, so you can try to use any of the methods described above that deal with radicals, e.g. trig. substitution, rationalizing substitutions, etc. One further method was encountered in class. It is based on the fact that sometimes algebraic expressions can be perfect squares despite what they look like originally. The basic trick is the formula

$$1+(A-B)^2=(A+B)^2 \quad \text{if it happens that} \quad AB=\frac{1}{4} \ \text{(why?)}$$

For instance, the following seemingly horrible function to integrate turns out to be quite harmless:

$$\sqrt{1+\left(3x^3-\frac{1}{12x^3}\right)^2}=\sqrt{\left(3x^3+\frac{1}{12x^3}\right)^2}=3x^3+\frac{1}{12x^3}.$$

Indeed, here we used $A=3x^3$ and $B=1/(12x^3)$, and noticed that $AB=1/4$.

If $f'(x)$ is discontinuous at one or both endpoints of $[a,b]$, turn the integrals into improper integrals and continue as usual when computing improper integrals. Go back to your classnotes and rewrite the solutions calculating arc length of a circle and surface area of a sphere as they should be written: with improper integrals.

4. Useful Formulas and Miscellaneous Facts

(1) The formula describing a circle of radius r as a function: $y=\pm\sqrt{r^2-x^2}$ or $x=\pm\sqrt{r^2-y^2}$.
(2) Quadratic formula for solving quadratic equations.
(3) Formulas for area and circumference of a circle, the volume and surface area of a sphere of radius r: πr^2, $2\pi r$, $\frac{4}{3}\pi r^3$, and $4\pi r^2$, respectively. How about the corresponding formulas for ellipse $x^2/a^2+y^2/b^2=1$ and the body resulting from revolving this ellipse about the x-axis? As an extra challenge, how about calculating the volume and surface area of a torus (#61 in §6.2)?
(4) Formulas for areas between curves and for volumes of solids of revolution.
(5) Fraction manipulations, exponential and logarithmic manipulations and formulas. Manipulations with the sigma notation: going back and forth between sigma notation and expanded notation.
(6) Exponential and logarithmic manipulations and formulas, e.g. $\frac{a^n}{b^n}=\left(\frac{a}{b}\right)^n$, $\ln a+\ln b=\ln ab$, $\ln a-\ln b=\ln\frac{a}{b}$, $c\cdot\ln a=\ln(a^c)$.
(7) Algebraic Formulas, e.g. $(A\pm B)^2=A^2\pm 2AB+B^2$, $(A+B)^3=A^3+3A^2B+3AB^2+B^3$, $(A-B)^3=A^3-3A^2B+3AB^2-B^3$.
(8) Differentiation Laws.

5. Problems for Review

The exam will be based on Howework and class problems. Review **all** homework problems, and all your classnotes and discussion notes. Such a thorough review should be enough to do well on the exam. If you want to give yourself a mock-exam, select 6 representative hard problems from various HW assignments, give yourself 1 hour and 20 minutes, and then compare your solutions to the HW solutions. If you didn't manage to do some problems, analyze for yourself what went wrong, which areas, concepts and theorems you should study in more depth, and if you ran out of time, think about how to manage your time better during the upcoming exam.

6. No Calculators during the Exam. Cheat Sheet and Studying for the Exam

No Calculators will be allowed during the exam. Anyone caught using a calculator will be disqualified from the exam.

For the exam, you are allowed to have a "cheat sheet" - *one page* of a regular 8.5×11 sheet. You can write whatever you wish there, under the following conditions:

- The whole cheat sheet must be **handwritten by your own hand**! No xeroxing, no copying, (and for that matter, no tearing pages from the textbook and pasting them onto your cheat sheet.)
- Any violation of these rules will disqualify your cheat sheet and may end in your own disqualification from the exam. I may decide to randomly check your cheat sheets, so let's play it fair and square. :)
- Don't be a **freakasaurus**! Start studying for the exam several days in advance, and prepare your cheat sheet at least 2 days in advance. This will give you enough time to become familiar with your cheat sheet and be able to use it more efficiently on the exam.
- **Do NOT overstudy on the day of the exam!! No sleeping the night before the exam due to cramming, or more than 3 hours of math study on the day of the exam is counterproductive! No kidding!**