Die-Hard Final Exam Review Exercises, Calculus II [1]

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Problem 1. **(Series Solutions to DE's)** Solve $y'' - 4y' + 4y = 0$ via the characteristic equation, and then separately via series. Compare your answers.

Problem 2. **(Resonance)** A car is driving on a bridge. The vibrations of the bridge are described by the continuous periodic function $F(t) = F_0 \cos(\omega_0 t)$. Assuming for simplicity that there is no damping ($c = 0$), describe the motion of the car caused by the bridge's vibrations via finding all solutions to the DE: $mx''(t) + kx(t) = F(t)$. When will the car's motion "go crazy", i.e. as time progresses the motion will become worse and worse? (*Answer:* When the two frequencies ω (of the car) and ω_0 (of the bridge) are **not** equal, then the solutions are $x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) + \frac{F_0}{m(\omega^2 - \omega_0^2)} \sin(\omega_0 t)$, which are "bad" when $\omega \approx \omega_0$, but $\omega \neq \omega_0$: in such a case, the motion of the car caused by the bridge's vibrations will be described by the sum of two simple harmonic motions with the second motion having a huge amplitude; so overall, the whole motion will be approximately a simple harmonic motion with a huge amplitude, rippled by the small "wiggles" of the other simple harmonic motion. Yet, as time progresses, the motion won't get any worse than that. However, when the two frequencies are equal: $\omega = \omega_0$, the solutions to the DE are $x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) + \frac{F_0}{2m\omega_0} t \sin(\omega_0 t)$. The the first two summands give a simple harmonic motion, but the last term is a "sine" function which wiggles between two lines through the origin: this second motion overwhelmes the first motion as time goes on and gives the "worse-and-worse" behavior as $t \to \infty$.)

Problem 3. **(Variation of Parameters)** Using variation of parameters solve the following two DEs: $y'' - 4y' + 4y = e^{2x} \tan x$, and $y'' + y = \cot x$. (*Note:* not all solutions will be written in the usual form: some functions may be represented by a integral.)

Problem 4. **(Logistic Model: Variations)**

(a) Without solving the DEs, find the inflection points of the solutions to the logistic model $dP/dt = 2P(1 - P/800)$. (*Answer:* All inflection points lie on the middle line $y = 400$ between the two equilibrium solutions $y = 0$ and $y = 800$.)

(b) Given a population with carrying capacity $K = 1000$, minimal survival level $m = 5$, and coefficient of initial exponential growth $k = 2$, find all functions representing the size of the population over time. (*Hint:* The DE is a variation of the basic logistic model: $dP/dt = 2P(1 - P/1000)(1 - 5/P)$, which after multiplying out $P(1-5/P) = P - 5$, becomes $dP/dt = 2(P-5)(1-P/1000)$, hence the equilibria are shifted to $P = 5$ and $P = 1000$. This DE is separable and can be solved correspondingly from scratch, but the sleek way of doing this problem is to reduce it to the basic logistic model by the substitution $Q(t) = P(t) - 5$, from where $dQ/dt = dP/dt$, and the DE becomes: $dQ/dt = 2Q(\frac{995-Q}{1000}) = ... = 1.99Q(1 - Q/995)$. From basic logistic model, $Q(t) = 995/(1 + Ae^{-1.99t})$, hence $P(t) = 5 + 995/(1 + Ae^{-1.99t})$ for any A, or $P(t) = 5$.)

Problem 5. **(Special Substitution into 2nd Order DE)** A small sphere of radius 1m is inside a big sphere of radius 2m: the two spheres are concentric, i.e. they have the same center. The temperatures of the surfaces of the two spheres are respectively 15°C (small sphere) and 25°C (big sphere). Find the temperature of a point exactly half way between the two spheres. (*Hint:* If $T(r)$ is the temperature of a point at distance r from the center, where $1 \leq r \leq 2$, then by a physics law the corresponding DE is $T''(r) + \frac{2}{r}T'(r) = 0$. This is NOT a separable DE since it has order 2. But we can make it into a separable DE by substituting $f(r) = T'(r)$: $f'(r) + \frac{2}{r}f(r) = 0$. Solving by separation of variables yields $f(r) = \frac{A}{r^2}$, and hence $T(r) = \int f(r)dr = -\frac{A}{r} + B$. To find the constants A and B, use the initial conditions $T(1) = 15$ and $T(2) = 25$: $A = 20$ and $B = 35$ so that $T(r) = -\frac{20}{r} + 35$. The wanted temperature $T(1.5) = 21.\overline{6}^\circ$C.)

Problem 6. **(Special 2nd Order Linear DE)** If $a, b, c > 0$ prove that any solution $y(x)$ to the DE $ay'' + by' + cy = 0$ dies out as time goes on, i.e. $\lim_{t\to\infty} y(x) = 0$. After solving the problem, if possible, relate your solutions to damped vibrations and give a physical explanation of this situation.

Problem 7. **(Orthogonal Trajectories)** Find the orthogonal trajectories to the family of curves $y = \frac{1}{x+k}$, where k varies over all real numbers. Draw a pictures of both families (the original family and its orthogonal trajectory family) and verify that indeed whenever curves from the two families intersect each other, they are orthogonal at these points of intersection.

Problem 8. **(Direction Fields and Integration Factors)** Sketch a direction field for $y' = y + xy$. Then solve the DE, draw representative solutions and verify that your drawn direction field is correct.

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[1] The exercises here are in general harder than what will appear on the Final Exam. Solving only these problems will NOT prepare you thoroughly for the final since not all topics are covered here. Hence, this list of problems is not intended to replace in any way the full review for the final exam. Apart from the hints and/or short answers here, solutions to these problems will NOT be posted on the web or distributed to students; thus, please, do NOT ask for solutions. Instead, try the problems on your own and if you have questions, ask the GSIs for help.