Review Topics for Midterm II in Calculus 1B

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1. Definitions and Basic Questions

Be able to **write** precise definitions for any of the following concepts (where appropriate: both in words and in symbols), to **give** examples of each definition, to **prove** that these definitions are satisfied in specific examples. Wherever appropriate, be able to **graph** examples for each definition.

(1) What is a *sequence*? How many ways of representing a sequence do we know? What is a *recursive* sequence and how does it differ from a sequences defined by a direct formula?
(2) What is a *convergent/divergent* sequence?
(3) What is the *limit* L of a sequence? Why do we need ε and N in the definition of $\lim\limits_{n\to\infty} a_n = L$ and what do they mean?
(4) What is an *increasing/decreasing* sequence? *monotonic* sequence?
(5) What is a *bounded* sequence? bounded *from below*? bounded *from above*?
(6) What is the relationship between *monotonic*, *bounded* and *convergent* sequence? Which of these property(ies) implies one of the other properties?
(7) What is a *subsequence* of a sequence? What is $\{a_{2n}\}$? What is the *odd-indexed* subsequence of a sequence? Can you find a formula for the index of every third term of a sequence $\{a_n\}$? How can you split the sequence $\{0, 1, 2, 0, 2, -2, 0, 3, 2, 0, 4, -2, 0, 5, 2, 0, 6, -2, ...\}$ into simpler subsequences, each of which has a limit, in order to argue formally that the whole sequence doesn't have a limit?
(8) What is the *"sequence" approach* versus the *"function approach"* in proving monotonicity and boundedness of a sequence?
(9) What is the *method of induction*? Why is it necessary in proving facts about recursive sequences? How do we find the limit of a recursive sequence? (Check out Example 12 in 11.1.)
(10) What are the *similarities and differences* between the following four sequences:

$$a_n = \frac{(-1)^n 2n}{3n+2}, b_n = \frac{(-1)^n 2^n n!}{5\cdot 8\cdot 11\cdots(3n+2)}, c_n = \sum_{k=1}^{n}\frac{(-1)^k 2k}{(3k+2)}, d_n = \sum_{k=1}^{n}\frac{(-1)^k 2^k k!}{5\cdot 8\cdot 11\cdots(3k+2)}?$$

What are the first 3 terms of each sequence? Can you determine if each of these sequences is monotine, bounded, convergent? If convergent, what is the limit?
(11) What is a *series*? a *partial sum* of a series? When do we say that a series *converges*? *diverges*?
(12) What is the difference between a *term of a sequence* and a *term of a series*? between the *sequence of terms* and the *sequence of partial sums* of a series? Which of latter two determines if a series converges?
(13) When given a series, which of the following do we have a *ready formula for*: the sequence of terms or the sequence of partial sums? Why?
(14) What is a *geometric* series? What makes a series be a geometric series - the first term or something else? What do we need to know in order to find the *sum* of the geometric series and how do we find this information by looking at the first several terms of a geometric series?
(15) What is the *harmonic* series? a p-series? What can you say about their convergence/divergence?
(16) What is the *telescoping method*? Which method of integration is it related to? Why is it called *telescoping* and to which series is it applicable? What questions does it answer when applied properly to a problem?
(17) What is the *remainder* R_n of a partial sum approximation? What is an *error estimate* of s_n given via the Integral Test (IT)?
(18) What is an *alternating* series? How do we know if a series is alternating? Is $\sum_{n\geq 1}(-1)^{3n+1}n$ alternating? How about $\sum_{n\geq 1}(-1)^{2n+1}n^2$, $\sum_{n\geq 1}(-1)^{n+1}\sin n$, $\sum_{n\geq 1}(-1)^{n-2}\cos(\pi n)$, $\sum_{n\geq 1} n^3\cos(\pi n)$?

(19) What is the *absolutely value series* (AVΣ) and what is its relation to the original series Σa_n? What is an *absolutely* convergent series? a *conditionally* convergent series? Can we have an absolutely convergent series which is not convergent in the ordinary sense? Why?

(20) What is a *rearrangement* of a series? Do these always converge to the same sum? Why? Which series exhibit behavior similar to that of finite sums: conditionally convergent, absolutely convergent, or divergent series? What qualifies here as "similar behavior"?

(21) Why are Ratio Test and Root Test powerless when the corresponding limits are equal to 1? How does one show convincingly that a test is inconclusive under certain conditions?

(22) What is a *power series*? a *power series* centered at a? the *coefficients* and the *variable* of a power series?

(23) What is the *radius* and *interval of convergence* of a power series? Relation to each other and to the center a?

(24) What does it mean to *represent* a function as a power series? What is the *term-by-term differentiation and integration* of a power series? Examples of power series expansions of $\ln x$, $\arctan x$, e^x, etc.

(25) What is a *Taylor series*, *Maclaurin series*, the *nth-degree Taylor polynomial of $f(x)$ at a*, the *remainder $R_n(x)$* of the Taylor series? Examples of Taylor series of basic functions: e^x, $\ln(1+x)$, $\cos x$, $\sin x$, $\arctan x$, $1/(1-x)$, $(1+x)^k$.

(26) What is *multiplication* and *division* of power series? How do we perform these? Are they always possible? Where does the resulting series converge?

(27) What is the *Binomial Theorem*? the *Binomial series*? Isn't the Binomial Theorem enough to cover all cases? Why do we also need Binomial series? For which kind of functions does the Binomial series help us find their Taylor series? Examples.

(28) What is an *application* of power series? Examples of approximations of π, e, $\ln 2$, etc. Examples of finding limits and approximating integrals using power series. Examples of approximating functions via several Taylor polynomials of successively higher degrees.

2. Theorems

Be able to **write** what each of the following theorems (laws, propositions, corollaries, etc.) says. Be sure to understand, distinguish and **state** the conditions (hypothesis) of each theorem and its conclusion. Be prepared to **give** examples for each theorem, and most importantly, to **apply** each theorem appropriately in problems. The latter means: decide which theorem to use, check (in writing!) that all conditions of your theorem are satisfied in the problem in question, and then state (in writing!) the conclusion of the theorem using the specifics of your problem.

(1) **Limit Theorems for sequences**. If $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist and are finite, then the following sequences also have limits:

$$\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n; \ \lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n; \ \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n},$$

where the last "division" theorem assumes in addition that $b_n \neq 0$ for all n and $\lim_{n\to\infty} b_n \neq 0$. If c is a constant, then as usual it can jump in front of the limit: $\lim\limits_{n\to\infty} c \cdot a_n = c \cdot \lim\limits_{n\to\infty} a_n$.

What happens in each of the above situations if $\lim a_n = -5$ and $\lim b_n$ doesn't exist? How about if $\lim a_n = 0$ and $\lim b_n$ doesn't exist? How about if $\lim a_n = \infty$ and $\lim b_n = 0$? Do we have any right to apply the above theorems in each of these cases? What do we do in each such case: how do we decide on what the limit is? (*Hint:* A theorem does not apply if its hypothesis (conditions) are not satisfied!)

(2) **Absolute Value Limit Theorem**: $\lim\limits_{n\to\infty} |a_n| = 0$ if and only if $\lim\limits_{n\to\infty} a_n = 0$.

Which direction of this theorem is used in practice? Can we replace in the theorem the limits 0 by another limit, say, 1? What can we conclude if $\lim_{n\to\infty} |a_n| \neq 0$?

(3) **Sandwich Theorem for sequences**. If $a_n \leq b_n \leq c_n$ for all n, and the limits of the policemen sequences are equal to each other: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$, then the middle (prisoner) sequence also converges to the same limit:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n.$$

As a version of this theorem, formulate what happens if a sequence a_n is larger than a sequence b_n and it is known that b_n diverges to ∞. (This could be referred to as the "∞ half-sandwich" theorem :)) Formulate the twin theorem when one of the sequences has limit $-\infty$: when and what can you conclude about the other sequence. Finally, suppose you have a sequence with positive terms which you want to show converges to 0; however, the given formula is very complex to deal with directly. What other sequence should you come up with in order to use a sandwich argument and conclude limit $= 0$?

(4) **Monotonic Bounded Sequence (MBT)**. If a_n is a monotonic and bounded sequence, then a_n converges.

Note that the **converse** is not true: If a_n converges, then a_n is bounded, but not necessarily monotonic. (Why? Examples? Counterexamples of sequences?) What do we use MBT for? Does it give us the limit of the sequence? What types of problems is MBT applied in?

(5) **Limit Sequence Thm.** If a sequence $\{a_n\}$ has limit L, then any subsequence of it also has limit L.

So, if it happens that two subsequences have different limits L_1 and L_2, how could the whole sequence converge? To what limit? If the whole sequence converged to some limit L, by our Limit Sequence theorem above, each subsequence must converge to the same L, but we have two specific subsequences in mind that don't abide to this rule: one converges to some L_1 and another converges to some another L_2. This is a blatant contradiction, hence the whole sequence has no chance to have a limit in this case. We formulate

(6) **Subsequence Limit Theorem 1.** If a sequence $\{a_n\}$ has two (or more) subsequences that converge to different limits, then the whole sequence $\{a_n\}$ does not have a limit, i.e. $\{a_n\}$ diverges. If a sequence $\{a_n\}$ has some subsequence which diverges, then the whole sequence $\{a_n\}$ also diverges.

(7) **Subsequence Limit Theorem 2.** If a sequence $\{a_n\}$ can be split into two (or three, or four, or finitely many) subsequences, each of which converges to the same limit L, then the whole sequence $\{a_n\}$ converges to that same common limit L.

(8) **Basic Example** of geometric sequences: $\{r^n\}$ converges/diverges as follows:

$$\lim_{n\to\infty} r^n \begin{cases} = 0 & \text{if } |r| < 1; \\ = 1 & \text{if } r = 1; \\ = \infty & \text{if } r > 1; \\ \nexists & \text{if } r \leq -1. \end{cases}$$

(9) **Finite Geometric Series Sum**: $a + ar + ar^2 + ar^3 + \cdots + ar^n = \dfrac{a(1-r^n)}{1-r}$ when $r \neq 1$.

(10) **Infinite Geometric Series Sum**: $a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots = \dfrac{a}{1-r}$ when $|r| < 1$. If $|r| \geq 1$, then the geometric series diverges.

(11) **The Harmonic Series Diverges**: $1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \cdots + \dfrac{1}{n} + \cdots = \infty$. The p-series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^p}$ converges for $p > 1$, and diverges for $p \leq 1$.

(12) **Theorem for convergence of sequences**: If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

(13) **Contrapositive Theorem (TD):** If $\lim_{n\to\infty} a_n \neq 0$ or does not exist, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

Thus, the condition $\lim_{n\to\infty} a_n = 0$ is *necessary* but **not sufficient** to guarantee convergence of a series. What does it mean that some condition X is *necessary but not sufficient* for Y to happen? How about *sufficient but not necessary* X? How about *necessary and sufficient* X? And finally, how about *neither necessary not sufficient* X? Provide in each of the 4 situations an example of corresponding X and Y related to convergence of series.

(14) **Telescoping Method** for series of the form $\displaystyle\sum_{n=1}^{\infty} \frac{ax+b}{(cn+d)(c(n+1)+d)}$, e.g. $\displaystyle\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \left(\frac{1}{1}-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots + \left(\frac{1}{n}-\frac{1}{n+1}\right) + \cdots = 1$ **because** the partial sums $s_n = 1 - \frac{1}{n+1}$ converge to 1, and hence the series itself converges with sum 1.

(15) **Theorems for Rearrangements(Riemann):** Let $\sum a_n$ be a series.

(a) If $\sum a_n$ is *absolutely* convergent with sum $s = \sum a_n$, then any rearrangement of $\sum a_n$ is also convergent with the same sum s.

(b) If $\sum a_n$ is *conditionally* convergent, then we can rearrange its terms in such a way that the new series converges to any (real) number we wish, as well as $\pm\infty$.

(c) If $\sum a_n$ diverges, there is nothing to talk about.

To summarize, absolutely convergent series behave very much like ordinary *finite* sums of numbers, while conditionally convergent series exhibit very different (and somewhat exotic) behavior.

(16) **Remainder Estimate for IT**. Let $\sum_{n=1}^{\infty} a_n$ with $a_n = f(n)$, where $f(x)$ is continuous, positive and decreasing function for $x \geq 1$. Suppose $\sum_{n=1}^{\infty} a_n$ converges with sum s (e.g. via IT). Then the remainder $R_n = s - s_n$ is estimated as follows:

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx.$$

Hence the error in approximating the sum s by the partial sum s_n ($s \approx s_n$) is at most $\int_n^{\infty} f(x)\,dx$, and the actual value of the sum s lies in the following interval:

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \leq s \leq s_n + \int_{n}^{\infty} f(x)\,dx.$$

Thus, to find out big n must be in order for s_n to approximate the sum s within error r, one solves for n the inequality $\displaystyle\int_n^{\infty} f(x)\,dx < r$: first turn the improper integral into a limit of proper integrals, integrate, find the limit, and then finally solve the inequality for n. The final answer should be of the form $n \geq N$.

(17) **Remainder Estimate for AT**. Let $\sum_{n=1}^{\infty}(-1)^n a_n$ be an alternating series which is convergent via AT, i.e. $a_n \geq 0$, $\{a_n\} \searrow$ for $n \gg 1$ and $\lim_{n\to\infty} a_n = 0$. Then the nth remainder $R_n = s - s_n$ is estimated by the $(n+1)$st term of the series:

$$|R_n| = |s - s_n| \leq a_{n+1}.$$

(18) **Remainder Estimate for CT.** Let $\sum_{n=1}^{\infty} a_n$ be convergent by Comparison Test with $\sum_{n=1}^{\infty} b_n$, i.e. $0 \leq a_n \leq b_n$ for $n \gg 1$ and $\sum_{n=1}^{\infty} b_n$ convergent. Then the nth remainder $R_n = s - s_n$ of $\sum_{n=1}^{\infty} a_n$ is estimated by the nth remainder T_n of $\sum_{n=1}^{\infty} b_n$. For instance, if $\sum_{n=1}^{\infty} b_n$ is convergent via IT, then the remainder R_n for $\sum_{n=1}^{\infty} a_n$ is bounded above by

$$R_n \leq T_n \leq \int_n^{\infty} f(x)\,dx.$$

If $\sum_{n=1}^{\infty} b_n$ is convergent via AT, then the remainder R_n for $\sum_{n=1}^{\infty} a_n$ is bounded above by

$$R_n \leq T_n \leq |b_{n+1}|.$$

(19) **Summary of Tests for Convergence/Divergence of Series**

Test	Conditions	Conclusion
1. *Geom. series* $\sum_{n=0}^{\infty} ar^n$	(a) If $\|r\| < 1$ (b) If $\|r\| \geq 1$	$\Rightarrow \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ converges $\Rightarrow \sum_{n=0}^{\infty} ar^n$ diverges
2. *P-series* $\sum_{n=0}^{\infty} \frac{1}{n^p}$	(a) If $p > 1$ (b) If $p \leq 1$	$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{n^p}$ converges $\Rightarrow \sum_{n=0}^{\infty} \frac{1}{n^p}$ diverges
3. *Comparison Test* (CT) $\sum_{n=0}^{\infty} a_n, \sum_{n=0}^{\infty} b_n$ (Note: $n \gg 0$ means "for all n from somewhere on")	$0 \leq a_n \leq b_n$ for $n \gg 0$ (a) If $\sum_{n=0}^{\infty} b_n$ converges (b) If $\sum_{n=0}^{\infty} a_n$ diverges	 $\Rightarrow \sum_{n=0}^{\infty} a_n$ converges $\Rightarrow \sum_{n=0}^{\infty} b_n$ diverges
4. *Limit Comparison Test* (LCT) $\sum_{n=0}^{\infty} a_n, \sum_{n=0}^{\infty} b_n$	$0 \leq a_n \leq b_n$ for $n \gg 0$ (a) If $\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$ (b) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0, \infty$ or $\nexists$	 $\Rightarrow \sum_{n=0}^{\infty} a_n$ & $\sum_{n=0}^{\infty} b_n$ behave similarly $\Rightarrow$ inconclusive
5. *Test for Divergence* (TD)	If $\lim_{n\to\infty} a_n \neq 0$ or $\nexists$	$\Rightarrow \sum_{n=0}^{\infty} a_n$ diverges
6. *Ratio Test* (RT) $\sum_{n=0}^{\infty} a_n$	(a) If $\lim_{n\to\infty} \left\| \frac{a_{n+1}}{a_n} \right\| = L < 1$ (b) If $\lim_{n\to\infty} \left\| \frac{a_{n+1}}{a_n} \right\| = L > 1$ or ∞ (c) If $\lim_{n\to\infty} \left\| \frac{a_{n+1}}{a_n} \right\| = L = 1$ or $\nexists$	$\Rightarrow \sum_{n=0}^{\infty} a_n$ converges (absolutely) $\Rightarrow \sum_{n=0}^{\infty} a_n$ diverges $\Rightarrow$ inconclusive
7. *Root Test* ($\sqrt{\ }$T) $\sum_{n=0}^{\infty} a_n$	(a) If $\lim_{n\to\infty} \sqrt[n]{\|a_n\|} = L < 1$ (b) If $\lim_{n\to\infty} \sqrt[n]{\|a_n\|} = L > 1$ or ∞ (c) If $\lim_{n\to\infty} \sqrt[n]{\|a_n\|} = L = 1$ or $\nexists$	$\Rightarrow \sum_{n=0}^{\infty} a_n$ converges (absolutely) $\Rightarrow \sum_{n=0}^{\infty} a_n$ diverges $\Rightarrow$ inconclusive
8. *Alternating Test* (AT) $\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} (-1)^n a_n$ If **all** are satisfied: 1. $a_n \geq 0$ for $n \gg 0$ (alternating) 2. $\{a_n\}$ is decreasing for $n \gg 0$ 3. $\lim_{n\to\infty} a_n = 0$	$\Rightarrow \sum_{n=0}^{\infty} (-1)^n a_n$ converges.
9. *Integral Test* (IT) $\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} a_n$ If **all** are satisfied: 1. a_n comes from $f(x)$ for $x \gg 1$ 2. $f(x)$ is decreasing for $x \gg 1$ 3. $f(x) \geq 0$ for $x \gg 1$	$\Rightarrow \int_1^{\infty} f(x)dx$ & $\sum_{n=0}^{\infty} a_n$ behave similarly.
10. *Absolute Convergence Test* (ACT) $\sum_{n=0}^{\infty} a_n$	If $\sum_{n=0}^{\infty} \|a_n\|$ converges	$\Rightarrow \sum_{n=0}^{\infty} a_n$ converges (absolutely)

(20) **Theorem for Convergence of Power Series:** Any power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ centered at a falls within one of the cases below:

(a) The power series converges only for $x = a$. Thus, $R = 0$ and the interval of convergence is $I = \{a\}$.

(b) The power series converges for all x. Thus, $R = \infty$ and the interval of convergence is $I = (-\infty, \infty)$.

(c) The power series converges for x within an interval I centered at a. Thus, there is some positive radius of convergence R, and the interval of convergence is $I = [a-R, a+R]$, or $I = [a-R, a+R)$, or $I = (a-R, a+R]$, or $I = (a-R, a+R)$. In other words, I always contains the open interval $(a-R, a+R)$, and it could also contain both, one, or none of the two endpoints $a \pm R$.

(21) **Theorem for Term-by-Term Differentiation and Integration of Power Series:** Let the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ have positive radius of convergence R, i.e. it defines a function

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots + c_n(x-a)^n + \cdots$$

for $x \in (a-R, a+R)$. Then this function $f(x)$ is continuous and differentiable, and its derivative and integral are calculated by performing term-by-term differentiation and integration on the power series:

(a) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots + nc_n(x-a)^{n-1} + \cdots;$

(b) $\displaystyle\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + c_3\frac{(x-a)^4}{4} + + \cdots + c_n\frac{(x-a)^{n+1}}{n+1} + \cdots.$

Further, the radii of convergence of the two new power series are also equal to R; however, the endpoints of the intervals of convergence may be different and require individual checks for each new series.

(22) **Theorem for Taylor Series Expansions:** If $f(x)$ has a power series expansion at a (radius $R > 0$):

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots + c_n(x-a)^n + \cdots ,$$

then the coefficients of this expansion are given by the formula $c_n = \dfrac{f^{(n)}(a)}{n!}$, $n = 0, 1, 2, 3, ...$ The *Maclaurin series* expansion is obtained by setting $a = 0$:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

(23) **Theorem for Remainders of Taylor Series Expansions:** Let

$$T_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n$$

be the nth-degree Taylor polynomial of $f(x)$ at a, and let $R_n(x) = f(x) - T_n(x)$ be the remainder of the Taylor series. If $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$, then $f(x)$ equals its Taylor series for $x \in (a-R, a+R)$.

(24) **Taylor's Inequality:** If there is number $M > 0$ which bounds all derivative functions of $f(x)$:

$$|f^{(n)}(x)| \leq M \ \forall n \text{ and } \forall x \in (a-R, a+R)$$

then the remainders

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \text{ for } x \in (a-R, a+R),$$

and $\lim_{n\to\infty} R_n(x) = 0$. Hence $f(x)$ equals its Taylor series for $x \in (a-R, a+R)$.

(25) **Table of Basic Taylor Series.** All functions here equal their Taylor series in the specified intervals.

Function	Taylor Series	Sum Notation	Interval of equality
$\dfrac{1}{1-x}$	$1+x+x^2+\cdots+x^n+\cdots$	$\sum_{n=0}^{\infty} x^n$	$(-1,1)$
$(1+x)^k$	$1+kx+\frac{k(k-1)}{2}+\frac{k(k-1)(k-2)}{6}x^3\cdots$	$\sum_{n=0}^{k}\binom{k}{n}x^n$	$(-1,1)$
e^x	$1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\cdots+\dfrac{x^n}{n!}+\cdots$	$\sum_{n=0}^{\infty}\frac{x^n}{n!}$	$(-\infty,+\infty)$
$\cos x$	$1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\dfrac{x^8}{8!}-\cdots$	$\sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n}}{(2n)!}$	$(-\infty,+\infty)$
$\sin x$	$x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+\dfrac{x^9}{9!}-\cdots$	$\sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{(2n+1)!}$	$(-\infty,+\infty)$
$\arctan x$	$x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+\dfrac{x^9}{9}-\cdots$	$\sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{2n+1}$	$[-1,+1]$
$\ln(1+x)$	$x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\dfrac{x^5}{5}-\cdots$	$\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{x^n}{n}$	$(-1,+1]$

(26) **Applications of Taylor Series for Enumeration of Fundamental Constants**

Constant and Series Enumeration	Taylor Series Used
$2=1+\dfrac{1}{2}+\dfrac{1}{2^2}+\dfrac{1}{2^3}+\cdots+\dfrac{1}{2^n}+\cdots=\sum_{n=0}^{\infty}\dfrac{1}{2^n}$	$\dfrac{1}{1-x},\, x=\dfrac{1}{2}$
$e=1+\dfrac{1}{1!}+\dfrac{1}{2!}+\dfrac{1}{3!}+\cdots+\dfrac{1}{n!}+\cdots=\sum_{n=0}^{\infty}\dfrac{1}{n!}$	$e^x,\ x=1$
$\cos 1=1-\dfrac{1}{2!}+\dfrac{1}{4!}-\dfrac{1}{6!}+\dfrac{1}{8!}-\cdots=\sum_{n=0}^{\infty}\dfrac{(-1)^n}{(2n)!}$	$\cos x,\ x=1$
$\sin 1=1-\dfrac{1}{3!}+\dfrac{1}{5!}-\dfrac{1}{7!}+\dfrac{1}{9!}-\cdots=\sum_{n=0}^{\infty}\dfrac{(-1)^n}{(2n+1)!}$	$\sin x,\ x=1$
$\pi=4\left(1-\dfrac{1}{3}+\dfrac{1}{5}-\dfrac{1}{7}+\dfrac{1}{9}-\cdots\right)=4\sum_{n=0}^{\infty}\dfrac{(-1)^n}{2n+1}$	$\arctan x,\ x=1$
$\ln 2=1-\dfrac{1}{2}+\dfrac{1}{3}-\dfrac{1}{4}+\dfrac{1}{5}-\cdots=\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n}$	$\ln(1+x),\ x=1$

3. Problem Solving Techniques

(1) **How do we find limits of sequences given by explicit formulas?** An explicit formula means a formula which gives us the power to calculate immediately any term of the sequence by plugging in the appropriate value of n.

(a) Say, $a_n=\dfrac{4n^2+2}{3n^2+10}$ for $n\geq 1$. Using the function approach here: $f(x)=\frac{4x^2+2}{3x^2+10}$, first find

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{4x^2+2}{3x^2+10}=\lim_{x\to\infty}\frac{x^2(4+\frac{2}{x^2})}{x^2(3+\frac{10}{x^2})}=\lim_{x\to\infty}\frac{(4+\frac{2}{x^2})}{(3+\frac{10}{x^2})}=\frac{4}{3}.$$

We recognized $f(x)$ as a fraction of two polynomials, so we factored the highest power of x from top and bottom, cancelled and then found the limit. We conclude that $\lim_{n\to\infty} a_n=\frac{4}{3}$.

We could have equivalently done this whole procedure *directly* on the sequence:

$$\lim_{n\to\infty} a_n=\lim_{n\to\infty}\frac{4n^2+2}{3n^2+10}=\lim_{n\to\infty}\frac{n^2(4+\frac{2}{n^2})}{n^2(3+\frac{10}{n^2})}=\lim_{n\to\infty}\frac{(4+\frac{2}{n^2})}{(3+\frac{10}{n^2})}=\frac{4}{3}.$$

Finally, we could have played "smart" and applied LH to $f(x)$:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{4x^2+2}{3x^2+10} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{(4x^2+2)'}{(3x^2+10)'} = \lim_{x\to\infty} \frac{8x}{6x} = \lim_{x\to\infty} \frac{4}{3} = \frac{4}{3}.$$

Each of the above three methods is acceptable in similar problems. Note that the function approach will work not only for fractions of polynomials, but also for a wide variety of cases, e.g. when trig., inverse trig. and log functions are involved.

(b) Say, $a_n = \dfrac{(n+2)!}{4^n}$. The function approach does not work when factorials are involved since no (reasonable) function can produce the sequence $n!$. So, we try a different approach. First, write a few terms of your sequence to see what is going on. Next, look at your sequence from "far away" - may be you can guess what its limit is. In our case, we have, roughly speaking, a factorial divided by an exponential. We know that factorials grow faster than exponentials, hence we predict that $\lim_{n\to\infty} a_n = \infty$.

The next step is to use the Sandwich Theorem: we have to cook policemen sequences which bound our sequence and converge to the same limit. However, in our example above we need only **one** policeman sequence b_n from below: we want $b_n \leq a_n$ and we want $b_n \to \infty$ (this will force $a_n \to \infty$). To this end, write a_n explicitly as follows:

$$a_n = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdots n\cdot (n+1)\cdot (n+2)}{4\cdot 4\cdot 4\cdot 4\cdot 4\cdot 4\cdots 4 \cdot 4 \cdot 4} = \left(\frac{1}{4}\right)\left(\frac{2}{4}\right)\left(\frac{3}{4}\right)\left(\frac{4}{4}\right)\left(\frac{5}{4}\right)\left(\frac{6}{4}\right)\cdots\left(\frac{n}{4}\right)\left(\frac{n+1}{4}\right)\left(\frac{n+2}{4}\right)$$

We notice that the first 3 fractions are < 1, the fourth fraction is $= 1$, and then, from the fifth fraction on, all other fractions are > 1; so no wonder that a_n keeps growing - we keep multiplying it my bigger and bigger fractions! To formalize this, leave alone the first (three) small fractions as well as the **last** fraction (we need that one to survive for the policeman!), and replace all middle fractions by 1. All in all, this **decreases** a_n:

$$a_n \geq \left(\frac{1}{4}\right)\left(\frac{2}{4}\right)\left(\frac{3}{4}\right)\cdot 1\cdot 1\cdots 1\cdot\left(\frac{n+2}{4}\right) = \frac{3}{128}(n+2).$$

Aha! Here is our policemen: set $b_n = \frac{3}{128}(n+2)$. We have shown $a_n \geq b_n$ (for $n \geq 3$, why?) But

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3}{128}(n+2) = \infty.$$

By the Sandwich Theorem, this forces $\lim_{n\to\infty} a_n = \infty$. □

Review the classnotes, workshop notes and HW for more examples of this sort. Note that you need two policemen if you are going to show, say, that $\lim_{n\to\infty} a_n = 8$: you need b_n and c_n such that $b_n \leq a_n \leq c_n$ and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 8$. Quite often, if you want to show $\lim_{n\to\infty} a_n = 0$, one of the policemen comes for free, e.g. it is often obvious that $0 \leq a_n$.

(c) What if $a_n = \dfrac{\sin(n+n^3-7)}{(-4)^n}$? The functions approach doesn't quite work here either because $(-4)^x$ doesn't always make sense (e.g. what is $(-4)^{1/2}$?) But let's look at the sequence from "far away" - we see that the numerator is just a sine function, so it can't go above 1 or below -1 no matter what n is! Moreover, the denominator goes "wild" when $n \to \infty$. I would bet that this sequence goes to 0. To show this, recall the absolute value limit theorem: it would apply perfectly here. So, set a new sequence which is the absolute value of a_n:

$$b_n = \frac{|\sin(n+n^3-7)|}{4^n}.$$

We got rid of the signs in the denominator, but we can't do much in the numerator since the sine functions oscillates between positive and negative, so we leave the absolute value sign in the numerator. Alright, now what? The Sandwich Theorem comes to the rescue once again:

we can cook up two policemen functions almost immediately - we first bound the numerator and then divide by the denominator:

$$0 \le |\sin(n+n^3-7)| \le 1 \;\Rightarrow\; 0 \le \frac{|\sin(n+n^3-7)|}{4^n} \le \frac{1}{4^n}.$$

Wonderful! Since both $\{0\}$ and $\{1/4^n\}$ converge to 0, then take their prisoner with them:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{|\sin(n+n^3-7)|}{4^n} = 0.$$

Finally, by the absolutely value limit theorem, this implies $\lim_{n\to\infty} a_n = 0$. □

There are plenty of similar examples from HW, so please, review them carefully. To summarize, there are essentially 3 tools in dealing with sequences given by explicit formulas: the function approach, the Sandwich Theorem, and the Absolute Value Limit Theorem. Each sequence requires careful thought: write out the first several terms, compare with any problem before, and try out different approaches until something works out.

(2) **How do we show that sequences are monotonic and/or bounded if given by an explicit formula?** First, write out the first several terms of your sequence to get a feeling for it: is it increasing/decreasing, bounded from above/below, etc. Keep in mind that some sequences can deceive you in the beginning, so make sure that you have written enough terms of the sequence to determine its behavior.

(a) Say, $a_n = \dfrac{4n^2+2}{3n^2+10}$ for $n \ge 1$ and suppose you think that the sequence is increasing. How do we show this? Again, there are two possible approaches.

(i) Try a function approach: $f(x) = \dfrac{4x^2+2}{3x^2+10}$, find the derivative $f'(x)$ and show that $f'(x) > 0$. Conclude that $f(x)$ increases, and hence forces $\{a_n\}$ also to increase.

(ii) Try a direct approach: set $a_n \overset{?}{\le} a_{n+1}$ and replace a_n and a_{n+1} by what they equal:

$$\frac{4n^2+2}{3n^2+10} \overset{?}{\le} \frac{4(n+1)^2+2}{3(n+1)^2+10}.$$

Cross-multiply and reduce this inequality to something obviously true. Note that expression on the RHS is indeed equal to a_{n+1}: we replaced n by $n+1$ in the original formula for a_n.

(b) What if we can't find/guess the "exact" upper/lower bound for a_n? How do we show that $\{a_n\}$ is bounded?

First of all, no one is asking you for "exact" bounds! If you find **one** number which is bigger than all a_n, then you have found **an** upper bound and you are done, and similarly for **a** lower bound. In the above example, if you write several terms of the sequence, you will notice that this sequence doesn't get very big or very small at all. So, you could conjecture, say, that $a_n < 30$ and try to show it directly:

$$a_n \overset{?}{<} 30 \;\Leftrightarrow\; \frac{4n^2+2}{3n^2+10} \overset{?}{<} 30.$$

Finish the calculation: reduce to something obviously true. It looks like I picked "30" out of the blue, and this is so indeed! I didn't want to bother with upper bounds that are "too close" to the sequence, so I picked something big enough that looked like a sure win. If you want to play smart, you could have argued that, since $\{a_n\}$ was shown to increase and to converge to 4/3, chances are that 4/3 is an upper bound (the "exact" upper bound, if you insist). So, you could have tried to show instead:

$$a_n \overset{?}{<} \frac{4}{3} \;\Leftrightarrow\; \frac{4n^2+2}{3n^2+10} \overset{?}{<} \frac{4}{3}.$$

I bet this wouldn't be terribly hard to finish. So, what are you waiting for?

Now, let's not forget that we could also be asked to find a lower bound. Well, once again, since the sequence was shown to be increasing, we have an automatic lower bound: the first term $a_1 = 6/13$. But we could pretend that we have just fallen from the moon and we didn't know that $\{a_n\}$ was increasing; in such a case, even a baby can see that 0 is a lower bound, or to "confuse the enemy", we could also say that -18.65963 is a lower bound, and we wouldn't be telling a lie.

(3) **How do we use the Monotonic Bounded Theorem when given a recursive formula?** For the sake of concreteness, let's use an exercise from §11.1: $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$. The problem asks to show that $\{a_n\}$ is increasing and bounded from above by 3, and then to find its limit.

(a) Let's show first that $a_n < 3$ for all n. We use induction.

Step 1. Is the first statement true? i.e. for $n = 1$? Sure, because $a_1 = \sqrt{2} < 3$.
Step 2. Suppose for a moment that the nth statement is true, i.e. $a_n < 3$.
Step 3. Using Step 2, we must show that the $(n + 1)$st statement is also true. Alright, we need to show that $a_{n+1} \overset{?}{<} 3$. But this is the same if we replace $a_{n+1} = \sqrt{2 + a_n}$:

$$\sqrt{2 + a_n} \overset{?}{<} 3 \ \Leftrightarrow\ 2 + a_n \overset{?}{<} 9 \ \Leftrightarrow\ a_n \overset{?}{<} 7.$$

Well, the last is certainly true since we have assumed something even stronger: $a_n < 3$. This shows that all of the previous inequalities are true, in particular, what we wanted: $a_{n+1} < 3$.

Steps 1-3 complete the proof by formal induction. We don't have to do anything now but conclude: "By induction, the above shows that *all* statements are true, i.e. $a_n < 3$ for all n." $\square$

Try to repeat the same discussion to show that $a_n < 2$ for all n - it will work the same way almost word-for-word! However, trying to show that $a_n < 1$ will fail: in which step and why?

(b) Now let's show that $\{a_n\}$ increases, i.e. $a_n \leq a_{n+1}$ for all n. We'll do this by induction.

Step 1. Is the first statement true? i.e. is it true that $a_1 \overset{?}{\leq} a_2$? Well, $a_1 = \sqrt{2}$, and $a_2 = \sqrt{2 + \sqrt{2}} > \sqrt{2 + 0} = \sqrt{2}$, so yes.
Step 2. Assume for a moment that the nth statement is true, i.e. $a_n \leq a_{n+1}$.
Step 3. Using Step 2, we will show that the $(n+1)$st statement is also true, i.e. $a_{n+1} \overset{?}{\leq} a_{n+2}$. But we know that $a_{n+1} = \sqrt{2 + a_n}$ and $a_{n+2} = \sqrt{2 + a_{n+1}}$. Replacing these in the inequality above, we get something equivalent:

$$\sqrt{2 + a_n} \overset{?}{\leq} \sqrt{2 + a_{n+1}} \ \Leftrightarrow\ 2 + a_n \overset{?}{\leq} 2 + a_{n+1} \ \Leftrightarrow\ a_n \overset{?}{\leq} a_{n+1}.$$

Well, the latter is certainly true by the assumption in Step 2. This completes the proof of Step 3, and ultimately, the proof of all stamements $a_n \leq a_{n+1}$. Thus, $\{a_n\}$ is increasing. $\square$

(c) A lot of information was accumulated about the sequence a_n: it is increasing and it is bounded from above by 3. But the former automatically implies that $\{a_n\}$ is bounded from below by its first term, $a_1 = \sqrt{2}$, i.e. $\sqrt{2} \leq a_n \leq 3$ for all n. Thus, $\{a_n\}$ is monotonic and bounded, hence by MBT: $\{a_n\}$ is convergent.

Now we have the right to name the limit of a_n, say, $L = \lim_{n\to\infty} a_n$, because we know that it exists and it is a finite number. To use this new-found knowledge, we go back to the very beginning of the problem: $a_{n+1} = \sqrt{2 + a_n}$. When we let $n \to \infty$ on both sides, we have

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2 + a_n}, \ \Rightarrow\ L = \sqrt{2 + L} \ \Rightarrow\ L^2 = 2 + L \ \Rightarrow\ L^2 - L - 2 = 0.$$

Solving the last quadratic equation yields $L = -1$ or $L = 2$. But $L = -1$ is impossible, since our sequence is bounded from below by $\sqrt{2}$ (even more so by 0!), so it cannot converge to a negative number. The only possibility left is $L = 2$. Therefore, $\lim_{n\to\infty} a_n = 2$. $\square$

(4) **How do we decide which test for series to apply?** There isn't a bulletproof recipe: each series requires particular attention and individual decision-making. Here are general strategies, which may or may not apply to the series you are considering.

Type of function, a_n	Tests to try in order of probable success
1. Rational or Algebraic (radicals,poly)	TD, CT or LCT (with p-series)
2. Exponentials (plus possibly poly,algebraic)	TD, RT, $\sqrt{\ }T$, CT or LCT (with geom. series), IT
3. Exponentials (plus some constants)	TD, CT or LCT (with geom. series)
4. Factorials (plus possibly poly, exponentials)	RT, CT or LCT, TD (Policemen Thm)
5. Alternating expression	AT, TD, ACT(+ other methods on AVS)
6. Lots of powers	$\sqrt{\ }T$, RT, TD
7. Simple rat'l functions	telescoping, TD, IT
8. Trigonometric (plus possibly some others)	TD, CT or LCT, ACT, AT, IT

(a) Let $\sum a_n$ be a series. Identify the type of function from which the terms a_n come from.

(b) Do **not** apply $\sqrt{\ }T$ if you have already found that RT is inconclusive: according to a theorem, if $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = 1$, then $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$, so $\sqrt{\ }T$ will also be inconclusive. As a general rule, do **not** apply RT or $\sqrt{\ }T$ when **only** polynomials or algebraic expressions are involved: in such cases, these two tests will inevitably yield limits 1, and hence will be inconclusive. However, as soon as there is a factorial involved (among other function), RT becomes a very plausible test, and as soon as there is an exponential involved, both RT and $\sqrt{\ }T$ become plausible tests.

(c) In series which are neither alternating nor their terms are only positive (e.g. the numerator is $\cos n$), you may try for absolute convergence coupled with CT.

(d) In using test involving limits (TD, LCT, RT, $\sqrt{\ }T$), keep in mind the following standard limits:

- $\lim\limits_{n\to\infty} \dfrac{\sin n}{n} = 1.$
- $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$ and $\lim\limits_{n\to\infty} \sqrt[n]{c} = 1$ for any constant $c > 0$.
- $\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n = e$. (Equivalently, $\lim\limits_{n\to\infty} \left(\dfrac{n}{1+n}\right)^n = \dfrac{1}{e}$. Why? What is $\lim\limits_{n\to\infty} \left(\dfrac{n+1}{n}\right)^{n+1}$?)

Be aware of variations of these limits where n is replaced by a linear function $an + b$: in some cases the limits remain the same, and in some cases the limits change. Investigate.

(e) If using AT or IT, you need to **verify all conditions in writing** before even starting to apply the test. In particular, in both tests there is a condition for decreasing terms a_n (in AT), or decreasing function $f(x)$ (in IT). Use whatever methods seem appropriate to show this in the specific problem: derivatives, direct approach ($a_n \geq a_{n+1}$), observation that the denominator obviously increases, etc. Do **not** use reasoning like "exponentials increase slower than factorials hence the fraction will be decreasing": even though this is true, we have not fully proved it; you can use such ideas just to start on the right track, but you need more solid standard explanations as suggested above when writing your solutions.

(5) **How do we find the radius and interval of convergence of a power series?** Let $\sum\limits_{n=0}^{\infty} c_n(x-a)^n$ be a power series. First note where the series is *centered at*: at a.

(a) To determine the *radius of convergence* R, imagine for a moment that x is a constant, and use RT or $\sqrt{\ }T$ - whichever seems easier. Setting the resulting limit to be less than 1, you end up with an equality of the form: $|x-a|<R$, e.g. $|x+2|<7$: in this case, the radius of convergence is $R=7$.

(b) To determine the *interval of convergence* I, solve the inequality above for x: $-R<x-a<R$, i.e. $a-R<x<a+R$ (in the example: $-7<x+2<7$, i.e. $-9<x<5$). Check separately for convergence of your original series at the endpoints of the interval: plug $x=a\pm R$ in the power series to obtain two ordinary series $\sum_{n=0}^{\infty} c_n R^n$ and $\sum_{n=0}^{\infty} c_n(-R)^n$, and use whatever methods necessary to determine convergence/divergence (note that RT or $\sqrt{\ }T$ will be useless here since you have exhausted their power earlier in the game when determining the radius R.)

(c) Summarize your findings: state your radius R and interval I, and draw a picture of the interval paying special attention to the endpoints. Check that the center a of your power series is indeed the center of the interval I (in the example, $a=-2$ is the center of $(-9,5)$.) Remember that *every power series* converges at least at its center a: if this isn't so in your series, check for silly computational errors.

(6) **How do we use geometric power series to find power series of other functions?** This is an **ad-hoc** method for finding power series representations of some functions $f(x)$. (The **general** method that works for all functions is the Taylor series approach.) The nice thing about the geometric power series ad-hoc approach is that it gives quickly the desired power series for $f(x)$ without the necessity of calculating all derivatives of $f(x)$ (as the Taylor series approach requires.) The drawback, as with all ad-hoc methods, is that it applies only to a very restricted class of functions: those that are related to geometric series either directly, or via a derivative or integral.

(a) *Functions directly related to geometric series:* $f(x)=\dfrac{p(x)}{bx+c}$ where $p(x)$ is a polynomial, and $bx+c$ is a linear function. We like $p(x)$ because *every polynomial is already a power series* in which almost all coefficients are 0 except, of course, the coefficients of $p(x)$, itself. We don't like the denominator $bx+c$ because power series don't have any denominators; so we turn $1/(bx+c)$ into a power series:

$$\begin{aligned}
\frac{p(x)}{bx+c} &= p(x)\cdot\frac{1}{c+bx} = p(x)\cdot\frac{1}{c(1-(-\frac{bx}{c}))}\\
&= \frac{p(x)}{c}\left(1+\left(-\frac{bx}{c}\right)+\left(-\frac{bx}{c}\right)^2+\left(-\frac{bx}{c}\right)^3+\cdots+\left(-\frac{bx}{c}\right)^n+\cdots\right)\\
&= \frac{p(x)}{c}\sum_{n=0}^{\infty}\left(-\frac{bx}{c}\right)^n = p(x)\cdot\sum_{n=0}^{\infty}\frac{(-1)^n b^n x^n}{c^{n+1}}.
\end{aligned}$$

In the above we used a geometric series with ratio $r=-\frac{bx}{c}$, so the power series expansion is valid only when $|-bx/c|<1$, i.e. $|x|<|c/b|$ (check it out!) To finish the problem, we have to bring $p(x)$ inside the sum, multiply out each term by it and regroup as necessary. For example, if $p(x) = x^2 - 2$ and $bx + c = -5x + 8$, following the above we obtain: ratio $r = 5x/8$,

$$\begin{aligned}
\frac{x^2-2}{-5x+8} &= (x^2-2)\cdot\sum_{n=0}^{\infty}\frac{5^n x^n}{8^{n+1}} = x^2\cdot\sum_{n=0}^{\infty}\frac{5^n x^n}{8^{n+1}} - 2\sum_{n=0}^{\infty}\frac{5^n x^n}{8^{n+1}} \\
&= \sum_{n=0}^{\infty}\frac{5^n x^{n+2}}{8^{n+1}} - \sum_{n=0}^{\infty}\frac{2\cdot 5^n x^n}{8^{n+1}} \text{ (shift } n \text{ to get } x^n \text{ everywhere)} \\
&= \sum_{n=2}^{\infty}\frac{5^{n-2} x^n}{8^{n-1}} - \sum_{n=0}^{\infty}\frac{2\cdot 5^n x^n}{8^{n+1}} \text{ (isolate initial terms for } n = 0, 1) \\
&= -\sum_{n=0}^{1}\frac{2\cdot 5^n x^n}{8^{n+1}} + \sum_{n=2}^{\infty}\left(\frac{5^{n-2}}{8^{n-1}} - \frac{2\cdot 5^n}{8^{n+1}}\right)x^n = -\frac{1}{4} - \frac{5}{32}x + \sum_{n=2}^{\infty}\frac{14\cdot 5^{n-2}}{8^{n+1}}x^n
\end{aligned}$$

Again, we don't forget that this power series expansion is valid only for $|5x/8| < 1$, i.e. $|x| < 8/5$. We needed to isolate the first two terms because they don't fit the general pattern of the terms in the sum (why?) The above calculations contain many algebraic and summation manipulations, so it is worth going through it carefully on your own. There is no need to memorize here any formulas: just understand how the method works and apply it individually to each similar problem.

(b) *Functions related to geometric series via derivative or integral:* logarithmic, arctan, reciprocals of powers of linear functions. Examples of such functions are $\ln(2x+7)$, $\arctan(2x+7)$, $\ln(4-x^3)$, $\arctan(x^3)$, $\frac{1}{(2x+7)^3}$, etc. There are essentially two ways of finding power series of these functions (apart from the Taylor series approach.)

- Recall (or rederive on the spot) the power series expansion of $\ln(1+x)$ and $\arctan x$, as well as their intervals of validity. Then you can substitute in them the new "variable expression".

$$\begin{aligned}
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}, \text{ for } x \in (-1, +1]; \\
\arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}, \text{ for } x \in [-1, +1].
\end{aligned}$$

It is clear how to obtain $\arctan(2x+7)$ and $\arctan(x^3)$ - substitute $2x+7$ and x^3 appropriately in the power series expansion for $\arctan x$:

$$\begin{aligned}
\arctan(2x+7) &= (2x+7) - \frac{(2x+7)^3}{3} + \frac{(2x+7)^5}{5} - \frac{(2x+7)^7}{7} + \frac{(2x+7)^9}{9} - \cdots \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{(2x+7)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n\frac{(x+\frac{7}{2})^{2n+1}}{2^{2n+1}(2n+1)} \text{ for } (2x+7) \in [-1, +1].
\end{aligned}$$

Note that this power series is centered at $-7/2$, and it is valid for $x \in [-4, -3]$ (why?)

$$\arctan(x^3) = x^3 - \frac{(x^3)^3}{3} + \frac{(x^3)^5}{5} - \frac{(x^3)^7}{7} + \frac{(x^3)^9}{9} - \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{(x^3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{6n+3}}{2n+1}$$

Note that this power series is centered at 0 and is valid for $x^3 \in [-1, +1]$, i.e. $x \in [-1, 1]$.

In the ln examples, we have to be more careful as to what exactly we substitute in $\ln(1+x)$, in other words, we need $\ln(1+\square)$. For $\ln(2x+7)$, factor 7 from $2x+7$, and then substitute

$2x/7$ in the power series expansion of $\ln(1+x)$:

$$\begin{aligned}\ln(2x+7) &= \ln\left(7\left(1+\frac{2x}{7}\right)\right) = \ln 7 + \ln\left(1+\frac{2x}{7}\right) = \ln 7 + \frac{(\frac{2x}{7})}{1} - \frac{(\frac{2x}{7})^2}{2} + \frac{(\frac{2x}{7})^3}{3} - \frac{(\frac{2x}{7})^4}{4} + \cdots \\ &= \ln 7 + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(\frac{2x}{7})^n}{n} = \ln 7 + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{7^n n}x^n\end{aligned}$$

for $\frac{2x}{7} \in (-1,+1]$, i.e. $x \in (-7/2, 7/2]$. Try $\ln(4-x^3)$ in the same vein.

- We can start from scratch, e.g. notice that $\ln(2x+7) = \int \frac{2}{2x+7}dx$ (why?), write the power expansion for $\frac{2}{2x+7}$, integrate it term by term, and find the constant C. Similarly, for $\arctan(2x+7) = \int \frac{2}{1+(2x+7)^2}dx$. Try both examples.

  In the case of $f(x) = \frac{1}{(2x+7)^3}$, integrating $f(x)$ twice yields a geometric series function, i.e.

$$\frac{1}{(2x+7)^3} = \frac{1}{8}\left(\frac{1}{2x+7}\right)'' = \frac{1}{56}\left(\frac{1}{1-(-\frac{2x}{7})}\right)'' = \frac{1}{56}\left(\sum_{n=0}^{\infty}\left(\frac{2x}{7}\right)^n\right)'' = \frac{1}{56}\left(\sum_{n=2}^{\infty}n(n-1)\left(\frac{2x}{7}\right)^{n-2}\right)$$

  Why does n start at 2 in the last summation? For what x is the power series expansion valid?

(c) *Rational Functions whose denominator has degree more than 1.* First we factor the denominator, split the fraction using partial fractions, and to each of the latter apply a geometric power series, and finally add up all the involved power series into one power series. When determining the radius of convergence of the final power series, we take the minimum of all involved radii of convergence. For example,

$$\begin{aligned}f(x) &= \frac{3}{x^2+x-2} = \frac{3}{(x-1)(x+2)} = \frac{1}{x-1} - \frac{1}{x+2} = -\frac{1}{1-x} - \frac{1}{2}\cdot\frac{1}{1-(-\frac{x}{2})} \\ &= -\sum_{n=0}^{\infty}x^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = -\sum_{n=0}^{\infty}x^n - \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}x^n = \sum_{n=0}^{\infty}\left(-1+\frac{(-1)^{n+1}}{2^n}\right)x^n\end{aligned}$$

Here the first power series expansion requires $|x| < 1$, while the second: $|x/2| < 1$, i.e. $|x| < 2$. The more restrictive requirement is $|x| < 1$, so that's where the final power series expansion for the original function is valid. □

(7) **How do we find the Taylor series of a function?** First, we agree on where the Taylor series must be centered at: usually this is part of the question, or else, it is assumed that it will be centered at 0 or at the most convenient for us point. Next, we find **all** derivatives of $f(x)$, or rather, we find a pattern for these derivatives. (Review §3.10 from Calculus I.) We then substitute in the formula for the coefficients of the Taylor series: $c_n = \frac{f^{(n)}(a)}{n!}$, where each derivative is evaluated at the center a. Make sure that what you get here are **numbers**, not functions, and don't forget about the factorial in the denominator. Finally, put all coefficients in the Taylor series formula:

$$\sum_{n=0}^{\infty}c_n(x-a)^n = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Make sure that you get a **power series** here in the "variable" $(x-a)$.

If the question asks for a specific Taylor polynomial, e.g. $T_4(x)$, then stop with the 4th derivative of $f(x)$, and produce the polynomial

$$T_4(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \frac{f''''(a)}{24}(x-a)^4.$$

Now, you don't have to perform Taylor calculations over and over again for similar functions. For instance, the Taylor series of $x^2 e^{-x^7}$ can be obtained by substituting $-x^7$ for x in the Taylor series of e^x, and then multiplying everything by x^2. Memorize the Taylor series of the basic functions given in a table earlier.

(8) **How do we decide for which x a function equals its Taylor expansion?** One way is to use the term-by-term differentiation and integration theorem: if you already know that a given Taylor series equals its function for some radius of convergence R around center a, then the derivative and the integral of this series are also Taylor series and they equal their own functions also on the interval $(a-R, a+R)$, and possibly at some of the two ends (to be checked separately).

Another more general and powerful way is to use Taylor's Inequality (mostly bonus stuff). Remember that the constant M you are looking for must be *universal*, i.e. it must be a bound for **all** derivative functions of $f(x)$. Once this is established, by Taylor's Inequality, we can conclude that the function does equal its Taylor series on the specificed interval.

(9) **Where can we apply known Taylor expansions of functions?**

(a) Special constants can be found by plugging some specific values of x into appropriate Taylor series. Recall how this was done for e, π, $\ln 2$, and think of possibly other constants that you can represent as infinite sums in a similar way. Conversely, Taylor series are useful in computing the sums of certain interesting series.

- In #38 on p.760, we ultimately want to find the sum $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}\cdot$ We follow the hints earlier.

  Let's find first $\displaystyle\sum_{n=1}^{\infty} nx^n,\ |x|<1$. This sum looks like the derivative of the geometric power series, but the power of x is wrong: we need x^{n-1}. So, let's force this power by factoring out x:

$$\sum_{n=1}^{\infty} nx^n \quad = \quad x\sum_{n=1}^{\infty} nx^{n-1} = x\left(\sum_{n=0}^{\infty} x^n\right)' = x\left(\frac{1}{1-x}\right)' = \frac{x}{(1-x)^2}\cdot$$

  Substituting in this formula $x = 1/2$, we find the next sum the problem asks for

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{\infty} n\frac{1}{2^n} = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = 2.$$

  Next, we set out to find $\displaystyle\sum_{n=2}^{\infty} n(n-1)x^n,\ |x|<1$. This sum looks like a second derivative of the geometric power series, but the power of x is wrong: we need x^{n-2}. So, let's force this power by factoring out x^2:

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-1)x^n \quad &= \quad x^2\sum_{n=2}^{\infty} n(n-1)x^{n-2} = x^2\left(\sum_{n=0}^{\infty} x^n\right)'' = x^2\left(\frac{1}{1-x}\right)'' \\ &= \quad x^2((1-x)^{-1})'' = x^2\cdot 2(1-x)^{-3} = \frac{2x^2}{(1-x)^3}\cdot\end{aligned}$$

OK, now the next sum they hint at is $\sum_{n=2}^{\infty} \frac{n^2-n}{2^n} = \sum_{n=2}^{\infty} n(n-1)\left(\frac{1}{2}\right)^n$. Obviously, we get exactly this sum if we plug in $x = 1/2$ in the previous sum. Thus,

$$\sum_{n=2}^{\infty} \frac{n^2-n}{2^n} = \frac{2\frac{1}{2^2}}{(1-\frac{1}{2})^3} = 4.$$

Finally, the sum we originally wanted to find is a sum of two series discussed above:

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \sum_{n=1}^{\infty} \frac{n^2-n}{2^n} + \sum_{n=1}^{\infty} \frac{n}{2^n} = 4 + 2 = 6.$$

- Evaluate the sum of $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}$. Which Taylor series might this have come from: the even factorials remind us of $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, which is valid everywhere, so no retrictions on x. Plugging in $x = \pi/6$ produces the desired series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

(b) Limits can be found by replacing one or more functions by their Taylor series. For example, below we could apply LH five times, but we can get away faster by using the Taylor series for $\sin x$:

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5} &= \lim_{x\to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} - \frac{x^9}{9!} + \cdots\right) - x + \frac{1}{6}x^3}{x^5} = \lim_{x\to 0} \frac{\frac{x^5}{5!} - \frac{x^7}{7!} - \frac{x^9}{9!} + \cdots}{x^5} \\
&= \lim_{x\to 0} \left(\frac{1}{5!} - \frac{x^2}{7!} - \frac{x^4}{9!} + \cdots\right) = \frac{1}{5!} - 0 + 0 - 0 + \cdots = \frac{1}{5!} = \frac{1}{120}.
\end{aligned}$$

(c) Integrals can be evaluated via series where "exact" integration in terms of ordinary functions is impossible. For example,

$$\begin{aligned}
\int \sin(x^2)dx &= \int \left(\frac{x^2}{1!} - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \frac{x^{18}}{9!} - \cdots\right) dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{2(2n-1)}}{(2n-1)!} \\
&= \frac{x^3}{3\cdot 1!} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{10\cdot 5!} - \frac{x^{15}}{15\cdot 7!} + \frac{x^{19}}{19\cdot 9!} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{4n-1}}{(4n-1)\cdot(2n-1)!}
\end{aligned}$$

(d) Convergence/divergence of ordinary series can be established via CT with the help of Taylor series: replace an involved function by its first or second Taylor polynomial to guess a new simpler series to compare with. For example, we are asked to test for convergence of $\sum_{n=1}^{\infty} \frac{\tan(1/n)}{n}$.

Consider the term $a_n = \frac{\sin(1/n)}{\cos(1/n)n}$, replace both $\sin x$ and $\cos x$ by their Taylor series, ignore all terms but the first, and cook up a new series to compare with:

$$a_n = \frac{\sin(1/n)}{\cos(1/n)n} = \frac{\frac{1}{n} - \frac{1}{n^3 3!} + \frac{1}{n^5 5!} + \cdots}{\left(1 - \frac{1}{n^2 2!} + \frac{1}{n^4 4!}\right)n} \sim b_n = \frac{\frac{1}{n}}{1\cdot n} = \frac{1}{n^2}.$$

Now we compare $\sum a_n$ with $\sum b_n$ via LCT:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(1/n)n}{\cos(1/n)} = \frac{\lim_{n\to\infty}\sin(1/n)n}{\lim_{n\to\infty}\cos(1/n)} = \lim_{n\to\infty} \sin(1/n)n.$$

The last is true because $\lim_{n\to\infty}\cos(1/n) = \cos(0) = 1$. Now we use LH, or resort to series again:

$$\lim_{n\to\infty} \sin(1/n)n = \lim_{n\to\infty}\left(\frac{1}{n} - \frac{1}{n^3 3!} + \frac{1}{n^5 5!} + \cdots\right)n = \lim_{n\to\infty}\left(1 - \frac{1}{n^2 3!} + \frac{1}{n^4 5!} + \cdots\right) = 1.$$

By LCT, $\sum a_n$ and $\sum b_n$ behave similarly; since $\sum b_n$ converges as 2-series, then $\sum a_n$ also converges.

(e) Multiplication and division of power series can be used to find Taylor series of more complicated functions.

(10) **Summary of Methods for Finding Taylor Series.** In lecture, we went through a long table, which had three columns: Method, Types of Functions, and Interval of Convergence. I will give here only the first column, and you are responsible to studying and filling in the other two columns.
- Basic power series.
- Substitution into basic power series.
- Algebraic manipulations and substitution into basic power series. Partial Fractions.
- TT$'$ or TT$\int$ on previous/basic power series.
- Taylor series (table method)
- Multiplication, division of previous power series.

4. Useful Formulas and Miscellaneous Facts

(1) Derivatives and integrals of special functions, e.g. trig., inverse trig., logarithmic, exponential, etc.
(2) Integration methods.
(3) Limits of special sequences and functions from before.
(4) All special values of trig. and inverse trig. functions.
(5) Quadratic formula for solving quadratic equations.
(6) Fraction manipulations: common denominators, splitting of fractions, cross-multiplication, splitting of radicals, manipulation of powers and factorials, etc.
(7) Representing a fraction as a sum of partial fractions.
(8) Exponential and logarithmic manipulations and formulas, e.g. $\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$, $\ln a + \ln b = \ln ab$, $\ln a - \ln b = \ln \frac{a}{b}$, $c \cdot \ln a = \ln(a^c)$.
(9) Algebraic Formulas, e.g. $(A \pm B)^2 = A^2 \pm 2AB + B^2$, $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$, $(A-B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$, $A^{n+m} = A^n \cdot A^m$, $(A^n)^m = A^{nm}$, $\sqrt[n]{A^m} = (\sqrt[n]{A})^m$, etc.

5. Review Problems

(1) All HW problems.
(2) All Quiz problems.
(3) All classnotes.
(4) All Review Section (except #60) in Chapter 11.
(5) For Bonus: as many as possible from Problems Plus Section in Chapter 11 (don't overkill yourselves here: this is only for those who like to hit their heads against a wall many times and eventually knock the wall down! :)

6. Cheat Sheet and Studying for the Exam

For the exam, you are allowed to have a "cheat sheet" - *one page* of a regular 8×11 sheet. You can write whatever you wish there, under the following conditions:

- The whole cheat sheet must be **handwritten by your own hand**! No xeroxing, no copying, (and for that matter, no tearing pages from the textbook and pasting them onto your cheat sheet.)
- Any violation of these rules will disqualify your cheat sheet and may end in your own disqualification from the midterm. I may decide to randomly check your cheat sheets, so let's play it fair and square. :)

- Don't be a **freakasaurus**! Start studying for the exam several days in advance, and prepare your cheat sheet at least 2 days in advance. This will give you enough time to become familiar with your cheat sheet and be able to use it more efficiently on the exam.
- **Do NOT overstudy on the day of the exam!! More than 3 hours of math study on the day of the exam is counterproductive! No kidding!**