Section 11.3, #32

Let $a_n = \frac{1}{n^{1+1/n}}$ and $b_n = \frac{1}{n}$. Note that a_n and b_n are both positive sequences, so the Limit Comparison Test applies.

So, we want to find $\lim_{n\to\infty} a_n/b_n = \lim_{n\to\infty} \dfrac{n}{n^{1+1/n}} = \lim_{n\to\infty} \dfrac{1}{n^{1/n}} = \lim_{n\to\infty} n^{-1/n}$. To do this, note

$$\lim_{n\to\infty} n^{-1/n} = \lim_{n\to\infty} e^{\ln n^{-1/n}} = \lim_{n\to\infty} e^{-1/n \ln n} = e^{\left[\lim_{n\to\infty}(-\frac{\ln n}{n})\right]}.$$

To find $\lim_{n\to\infty}(-\dfrac{\ln n}{n})$, use L'Hospital's rule since $\lim_{n\to\infty} \ln n = \infty$ and $\lim_{n\to\infty} n = \infty$.

$$\lim_{n\to\infty}(-\frac{\ln n}{n}) = \lim_{n\to\infty}(-\frac{1/n}{1}) = \lim_{n\to\infty}(-1/n) = 0.$$

Therefore,

$$\lim_{n\to\infty} n^{-1/n} = e^{\left[\lim_{n\to\infty}(-\frac{\ln n}{n})\right]} = e^0 = 1.$$

Since $\lim_{n\to\infty} a_n/b_n = 1 > 0$, the Limit Comparison Test says that $\sum_{n=0}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} b_n$ converges. But $\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} 1/n$ is just the Harmonic Series, which is known to diverge. Thus,

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{1}{n^{1+1/n}}$$

must also diverge.