Calculus 1A, Spring '12

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§1. Some Continuity Laws

Hypothesis for all continuity theorems below:

If $f(x)$ and $g(x)$ are *continuous* at $x = a$, i.e. $\lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to a} g(x) = g(a)$, then

#	Theorem Name	Conclusion	Follows from
1	Continuity of Sum	$f(x)+g(x)$ is also continuous at $x = a$	LL for sum
2	Continuity of Difference	$f(x)-g(x)$ is also continuous at $x = a$	LL for difference
3	Continuity of Product	$f(x)g(x)$ is also continuous at $x = a$	LL for product
4	Continuity of Ratio ($g(a) \neq 0$)	$f(x)/g(x)$ is also continuous at $x = a$	LL for ratio
5	Continuity Respects Constants	$c \cdot f(x)$ is also continuous at $x = a$	LL for jumping constants
6	Continuity of Composition ($h(x)$ is continuous at $b = f(a)$)	$h(f(x))$ is also continuous at $x = a$	LL for composition

Note: All *Continuity Laws (CLs)* follow from the corresponding *Limit Laws (LLs)*. The CLs above allow us to perform algebraic operations (and compositions) on continuous functions. Thus, we can construct more complex continuous functions from simpler continuous functions. To do this, we need to have a starting collection of

§2. Basic Continuous Functions

All of the following types of functions below are *continuous* on their respective domains of definition:

#	Function	Algebraic Formula and Conclusion	Follows from
1	Constants	c continuous at $\forall x$	LL for constants
2	Linear	$ax + b$ continuous at $\forall x$	LL for linear fn's
3	Quadratic	$ax^2 + bx + c$ continuous at $\forall x$	LL for quadratic fn's
4	Power	x^n continuous at $\forall x$, $\forall n = 1, 2, 3, ...$	LL for powers
5	Polynomial	$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ continuous at $\forall x$	LL for polynomials
6	Rational	$\dfrac{f(x)}{g(x)}$ continuous where $g(x) \neq 0$ ($f(x), g(x)$ - poly's)	LL for ratio, CL for poly's
7	Root	$\sqrt[n]{x}$ continuous at $\forall x$ where defined	LL for roots
8	Exponential	a^x continuous at $\forall x$ ($a > 0$)	LL for exponentials
9	Logarithmic	$\log_a x,\ \ln x$ continuous at $\forall x > 0$ ($a > 0$)	LL for logarithmics
10	Trigono-metric	$\sin x,\ \cos x$ continuous at $\forall x$; $\tan x,\ \cot x$ cont. on domain: $\tan x$: $x \neq \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2 \ldots, (2n+1)\pi/2$, $\cot x$: $x \neq 0, \pm\pi, \pm 2\pi, \pm 3\pi \ldots, n\pi,\ n \in \mathbb{Z}$	LL for trig. fn's
11	Inverse Trigono-metric	arcsin$x$, arccos$x$, arctan$x$, arccot$x$ continuous on domain: arcsin$x : [-1, 1] \to [-\pi/2, \pi/2]$, arccos$x : [-1, 1] \to [0, \pi]$, arctan$x : \mathbb{R} \to (-\pi/2, \pi/2)$, arccot$x : \mathbb{R} \to (0, \pi)$	LL for inverse trig. fn's

§3. Summary of Limit–Definition Types of Goals and Answers

Here are all 9 possible types of limits (3 possible goals, and 3 possible answers) for

$$\lim_{x \to \Box_1} f(x) = \Box_2$$

Limit $\Box_2$	Goal for $f(x)$	Want for $f(x)$:	$x \to \Box_1$	Answer for x	Expect for x:
L	ϵ-goal around L	$L - \epsilon < f(x) < L + \epsilon$	$x \to a$	δ-interval around a	$a - \delta < x < a + \delta$
$+\infty$	M-goal, $M > 0$	$f(x) > M$	$x \to +\infty$	N-answer, $N > 0$	$x > N$
$-\infty$	M-goal, $M < 0$	$f(x) < M$	$x \to -\infty$	N-answer, $N < 0$	$x < N$

Can you think of 9 concrete examples of limits featuring the 9 different types of limits in the table? For instance, when $\Box_2 = -\infty$ and $\Box_1 = +\infty$, we have $\lim_{x \to +\infty} -x^2 = -\infty$.

§4. Infinite Limit Laws (∞LLs)

In the infinite limit laws below (∞LLs), an expression like "$(-\infty) + (-\infty) = -\infty$" does NOT have a meaning on its own, except when *in context*, i.e., it refers only to the following situation and to nothing else:

Theorem ($\infty LL+$). *"If for two functions $f(x)$ and $g(x)$ we know that $\lim_{x \to \Box} f(x) = -\infty$ and $\lim_{x \to \Box} g(x) = -\infty$, then $f(x) + g(x)$ also has a limit when $x \to \Box$, and this limit is $\lim_{x \to \Box} (f(x) + g(x)) = -\infty$."* (Here $x \to \Box$ can mean anything of the following: $x \to a$, $x \to \infty$, or $x \to -\infty$.)

The table of ∞LLs on the next page us to treat the symbols $+\infty$ and $-\infty$ **almost** like numbers: you can plug them into expressions, just like numbers, and get a meaningful answer... EXCEPT when an **indeterminate** is achieved, e.g., $\infty - \infty$, $0 \cdot \infty$, and $\frac{\pm\infty}{\pm\infty}$ make no sense and cannot be given any uniform meaning. In fact, these indeterminate cases distinguish $\pm\infty$ from ordinary (finite) numbers and warn us to be careful when using ∞LLs. We will discuss later in the course how to deal with indeterminates like ∞/∞, $0/0$, and 1^∞: there is a very powerful (but dangerous!) tool, which we will develop to conquer them. :)

TURN OVER FOR TABLE OF INFINITE LIMIT LAWS.

Name	∞LL: Formula	Example
1. addition	$\infty+\infty=\infty$	$\lim\limits_{x\to\infty}(x+x^2)=\lim\limits_{x\to\infty}x+\lim\limits_{x\to\infty}x^2=$ “$\infty+\infty$” $\overset{\infty\mathrm{LL}}{=}\infty$
$\infty LL+$	$(-\infty)+(-\infty)=-\infty$	$\lim\limits_{x\to\infty}(-x-x^2)=\lim\limits_{x\to\infty}(-x)+\lim\limits_{x\to\infty}(-x^2)=$ “$(-\infty)+(-\infty)$” $\overset{\infty\mathrm{LL}}{=}-\infty$
	$\infty-\infty$ undefined	**Never use ∞LLs in such cases.**
2. multipl.	$\infty\cdot\infty=\infty$	$\lim\limits_{x\to\infty}x^2=\lim\limits_{x\to\infty}x\cdot\lim\limits_{x\to\infty}x=\infty\cdot\infty\overset{\infty\mathrm{LL}}{=}\infty$
$\infty LL*$	$(-\infty)\cdot(-\infty)=\infty$	$\lim\limits_{x\to\infty}(-x)^2=\lim\limits_{x\to\infty}(-x)\cdot\lim\limits_{x\to\infty}(-x)=$ “$(-\infty)\cdot(-\infty)$” $\overset{\infty\mathrm{LL}}{=}\infty$
	$\infty\cdot(-\infty)=-\infty$	$\lim\limits_{x\to\infty}-x^2=\lim\limits_{x\to\infty}x\cdot\lim\limits_{x\to\infty}(-x)=$ “$\infty\cdot(-\infty)$” $\overset{\infty\mathrm{LL}}{=}-\infty$
3. multipl.	$c\cdot\infty=+\infty$ if $c>0$	$\lim\limits_{x\to\infty}(2x)=2\lim\limits_{x\to\infty}x=$ “$2\cdot\infty$” $\overset{\infty\mathrm{LL}}{=}\infty$
by c	$c\cdot\infty=-\infty$ if $c<0$	$\lim\limits_{x\to\infty}(-2x)=-2\lim\limits_{x\to\infty}x=$ “$-2\cdot\infty$” $\overset{\infty\mathrm{LL}}{=}-\infty$
$\infty LL{*}c$	$0\cdot\infty$ undefined	**Never use ∞LLs in such cases.**
4. adding c	$c+\infty=+\infty\ \forall c$	$\lim\limits_{x\to\infty}(-2+x)=-2+\lim\limits_{x\to\infty}x=$ “$-2+\infty$” $\overset{\infty\mathrm{LL}}{=}\infty$
$\infty LL+c$	$c-\infty=-\infty\ \forall c$	$\lim\limits_{x\to\infty}(-2-x)=-2-\lim\limits_{x\to\infty}x=$ “$-2-\infty$” $\overset{\infty\mathrm{LL}}{=}-\infty$
5. div. by ∞	$\frac{c}{+\infty}=0\ \forall c$	$\lim\limits_{x\to\infty}\frac{2}{x}=\frac{2}{\lim\limits_{x\to\infty}x}=$ “$\frac{2}{\infty}$” $\overset{\infty\mathrm{LL}}{=}0$
$\infty LL{\div}\infty$	$\frac{c}{-\infty}=0\ \forall c$	$\lim\limits_{x\to\infty}\frac{2}{-x}=\frac{2}{\lim\limits_{x\to\infty}(-x)}=$ “$\frac{2}{-\infty}$” $\overset{\infty\mathrm{LL}}{=}0$
6. div. by ±0	$\frac{c}{0^+}=+\infty\ \forall c>0$	$\lim\limits_{x\to0}\frac{2}{x^2}=$ “$\frac{2}{0^+}$” $\overset{\infty\mathrm{LL}}{=}+\infty$, $\lim\limits_{x\to0}\frac{-2}{x^2}=$ “$\frac{-2}{0^+}$” $\overset{\infty\mathrm{LL}}{=}-\infty$
$\infty LL\div 0$	$\frac{c}{0^-}=-\infty\ \forall c>0$	$\lim\limits_{x\to0}\frac{2}{-x^2}=$ “$\frac{2}{0^-}$” $\overset{\infty\mathrm{LL}}{=}-\infty$, $\lim\limits_{x\to0}\frac{-2}{-x^2}=$ “$\frac{-2}{0^-}$” $\overset{\infty\mathrm{LL}}{=}+\infty$
	$\frac{0}{0},\ \frac{c}{0}$ undefined $\forall c$	**Never use ∞LLs in such cases.**
7.basic ∞LL	$\frac{1}{\infty}=0,\ \frac{1}{-\infty}=0$	$\lim\limits_{x\to\infty}\frac{1}{x}=$ “$\frac{1}{\infty}$” $=0$, $\lim\limits_{x\to-\infty}\frac{1}{x}=$ “$\frac{1}{-\infty}$” $=0$
$\infty LL\frac{1}{x}$	$\frac{1}{0^+}=+\infty,\ \frac{1}{0^-}=-\infty$	$\lim\limits_{x\to0^+}\frac{1}{x}=$ “$\frac{1}{0^+}$” $+\infty$, $\lim\limits_{x\to0^-}\frac{1}{x}=$ “$\frac{1}{0^-}$” $=-\infty$, $\lim\limits_{x\to0}\frac{1}{x}$ **does not exist.**
$\infty LL\,x,\sqrt{x}$	$\sqrt{\infty}=\infty$	$\lim\limits_{x\to\infty}x=\infty,\ \lim\limits_{x\to-\infty}x=-\infty,\ \lim\limits_{x\to\infty}\sqrt{x}=$ “$\sqrt{\infty}$” $=\infty$
8. exponen-	$c^{\infty}=+\infty,\ c^{-\infty}=0$ if $c>1$	$\lim\limits_{x\to\infty}2^x=$ “2^{∞}” $\overset{\infty\mathrm{LL}}{=}\infty$; $\lim\limits_{x\to-\infty}e^x=$ “$e^{-\infty}$” $=$ “$\frac{1}{e^{\infty}}$” $=$ “$\frac{1}{\infty}$” $\overset{\infty\mathrm{LL}}{=}0$
tiation by ∞	$c^{\infty}=0,\ c^{-\infty}=\infty$ if $0\le c<1$	$\lim\limits_{x\to\infty}\left(\frac{1}{3}\right)^x=$ “$\left(\frac{1}{3}\right)^{\infty}$” $\overset{\infty\mathrm{LL}}{=}0$; $\lim\limits_{x\to-\infty}0.9^x=$ “$0.9^{-\infty}$” $=$ “$\left(\frac{1}{0.9}\right)^{\infty}$” $\overset{\infty\mathrm{LL}}{=}\infty$
∞LLc^{∞}	1^{∞} undefined	**Never use ∞LLs in such cases.**