Calculus 1A, Spring'09
Instructor: with Professor Zvezdelina Stankova

§1. Continuity Laws

Hypothesis for all continuity theorems below:
If $f(x)$ and $g(x)$ are *continuous* at $x = a$, i.e. $\lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to a} g(x) = g(a)$, then

#	Theorem Name	Conclusion	Follows from
1	Continuity of Sum	$f(x)+g(x)$ is also continuous at $x = a$	LL for sum
2	Continuity of Difference	$f(x)-g(x)$ is also continuous at $x = a$	LL for difference
3	Continuity of Product	$f(x)g(x)$ is also continuous at $x = a$	LL for product
4	Continuity of Ratio ($g(a) \neq 0$)	$f(x)/g(x)$ is also continuous at $x = a$	LL for ratio
5	Continuity Respects Constants	$c \cdot f(x)$ is also continuous at $x = a$	LL for jumping constants
6	Continuity of Composition ($h(x)$ is continuous at $b = f(a)$)	$h(f(x))$ is also continuous at $x = a$	LL for composition

Note: All *Continuity Laws (CLs)* follow from the corresponding *Limit Laws (LLs)*. The CLs above allow us to perform algebraic operations (and compositions) on continuous functions. Thus, we can construct more complex continuous functions from simpler continuous functions. To do this, we need to have a starting collection of

§2. Basic Continuous Functions

All of the following types of functions below are *continuous* on their respective domains of definition:

#	Function	Algebraic Formula and Conclusion	Follows from
1	Constants	c continuous at $\forall x$	LL for constants
2	Linear	$ax + b$ continuous at $\forall x$	LL for linear fn's
3	Quadratic	$ax^2 + bx + c$ continuous at $\forall x$	LL for quadratic fn's
4	Power	x^n continuous at $\forall x$, $\forall n = 1, 2, 3, ...$	LL for powers
5	Polynomial	$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ continuous at $\forall x$	LL for polynomials
6	Rational	$\dfrac{f(x)}{g(x)}$ continuous where $g(x) \neq 0$ ($f(x), g(x)$ - poly's)	LL for ratio, CL for poly's
7	Root	$\sqrt[n]{x}$ continuous at $\forall x$ where defined	LL for roots
8	Exponential	a^x continuous at $\forall x$ ($a > 0$)	LL for exponentials
9	Logarithmic	$\log_a x,\ \ln x$ continuous at $\forall x > 0$ ($a > 0$)	LL for logarithmics
10	Trigono-metric	$\sin x,\ \cos x$ continuous at $\forall x$; $\tan x,\ \cot x$ cont. on domain: $\tan x$: $x \neq \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2 \ldots, (2n+1)\pi/2$, $\cot x$: $x \neq 0, \pm\pi, \pm 2\pi, \pm 3\pi \ldots, n\pi,\ n \in \mathbb{Z}$	LL for trig. fn's
11	Inverse Trigono-metric	$\arcsin x,\ \arccos x,\ \arctan x,\ \operatorname{arccot} x$ continuous on domain: $\arcsin x : [-1, 1] \to [-\pi/2, \pi/2]$, $\arccos x : [-1, 1] \to [0, \pi]$, $\arctan x : \mathbb{R} \to (-\pi/2, \pi/2)$, $\operatorname{arccot} x : \mathbb{R} \to (0, \pi)$	LL for inverse trig. fn's

§3. Infinite Limit Laws

Note: In the infinite limit laws below (∞-LL), an expression like "$(-\infty)+(-\infty)=-\infty$" does not have a meaning on its own, except *in context*, i.e. it refers only to the following situation and to nothing else:

Theorem. *"If for two functions $f(x)$ and $g(x)$ we know that $\lim\limits_{x\to\square} f(x)=-\infty$ and $\lim\limits_{x\to\square} g(x)=-\infty$, then $f(x)+g(x)$ also has a limit when $x\to\square$, and this limit is $\lim\limits_{x\to\square}(f(x)+g(x))=-\infty$."*
Here $x\to\square$ can mean anything of the following: $x\to a$, $x\to\infty$ or $x\to-\infty$.

Name	∞-LL: Formula	Example
1. addition	$\infty+\infty=\infty$	$\lim\limits_{x\to\infty}(x+x^2)=\lim\limits_{x\to\infty}x+\lim\limits_{x\to\infty}x^2=\infty+\infty\overset{\infty\text{LL}}{=}\infty$
	$(-\infty)+(-\infty)=-\infty$	$\lim\limits_{x\to\infty}(-x-x^2)=\lim\limits_{x\to\infty}(-x)+\lim\limits_{x\to\infty}(-x^2)=(-\infty)+(-\infty)\overset{\infty\text{LL}}{=}-\infty$
	$\infty-\infty$ undefined	**Never use ∞-LLs in such cases.**
2. multipli- cation	$\infty\cdot\infty=\infty$	$\lim\limits_{x\to\infty}x^2=\lim\limits_{x\to\infty}x\cdot\lim\limits_{x\to\infty}x=\infty\cdot\infty\overset{\infty\text{LL}}{=}\infty$
	$(-\infty)\cdot(-\infty)=-\infty$	$\lim\limits_{x\to\infty}(-x)^2=\lim\limits_{x\to\infty}(-x)\cdot\lim\limits_{x\to\infty}(-x)=(-\infty)\cdot(-\infty)\overset{\infty\text{LL}}{=}\infty$
	$\infty\cdot(-\infty)=\infty$	$\lim\limits_{x\to\infty}-x^2=\lim\limits_{x\to\infty}x\cdot\lim\limits_{x\to\infty}(-x)=\infty\cdot(-\infty)\overset{\infty\text{LL}}{=}-\infty$
3. multipli- cation by constant	$c\cdot\infty=+\infty$ if $c>0$	$\lim\limits_{x\to\infty}(2x)=2\lim\limits_{x\to\infty}x=2\cdot\infty\overset{\infty\text{LL}}{=}\infty$
	$c\cdot\infty=-\infty$ if $c<0$	$\lim\limits_{x\to\infty}(-2x)=-2\lim\limits_{x\to\infty}x=-2\cdot\infty\overset{\infty\text{LL}}{=}-\infty$
	$0\cdot\infty$ undefined	**Never use ∞-LLs in such cases.**
4. addition by constant	$c+\infty=+\infty\ \forall c$	$\lim\limits_{x\to\infty}(-2+x)=-2+\lim\limits_{x\to\infty}x=-2+\infty\overset{\infty\text{LL}}{=}\infty$
	$c-\infty=-\infty\ \forall c$	$\lim\limits_{x\to\infty}(-2-x)=-2-\lim\limits_{x\to\infty}x=-2-\infty\overset{\infty\text{LL}}{=}-\infty$
5. division by $\pm\infty$	$\dfrac{c}{+\infty}=0\ \forall c$	$\lim\limits_{x\to\infty}\dfrac{2}{x}=\dfrac{2}{\lim\limits_{x\to\infty}x}=\dfrac{2}{\infty}\overset{\infty\text{LL}}{=}0$
	$\dfrac{c}{-\infty}=0\ \forall c$	$\lim\limits_{x\to\infty}\dfrac{2}{-x}=\dfrac{2}{\lim\limits_{x\to\infty}(-x)}=\dfrac{2}{\infty}\overset{\infty\text{LL}}{=}0$
6. division by $0^{\pm}$	$\dfrac{c}{0^+}=+\infty\ \forall c>0$	$\lim\limits_{x\to0}\dfrac{2}{x^2}=\dfrac{2}{0^+}\overset{\infty\text{LL}}{=}+\infty$, $\lim\limits_{x\to0}\dfrac{-2}{x^2}=\dfrac{-2}{0^+}\overset{\infty\text{LL}}{=}-\infty$
	$\dfrac{c}{0^-}=-\infty\ \forall c>0$	$\lim\limits_{x\to0}\dfrac{2}{-x^2}=\dfrac{2}{0^-}\overset{\infty\text{LL}}{=}-\infty$, $\lim\limits_{x\to0}\dfrac{-2}{-x^2}=\dfrac{-2}{0^-}\overset{\infty\text{LL}}{=}+\infty$
	$\dfrac{0}{0}$, $\dfrac{c}{0}$ undefined $\forall c$	**Never use ∞-LLs in such cases.**
7. basic cases	$\lim\limits_{x\to\infty}\dfrac{1}{x}=0$, $\lim\limits_{x\to-\infty}\dfrac{1}{x}=0$	
	$\lim\limits_{x\to\infty}x=\infty$, $\lim\limits_{x\to-\infty}x=-\infty$	
	$\lim\limits_{x\to0^+}\dfrac{1}{x}=+\infty$, $\lim\limits_{x\to0^-}\dfrac{1}{x}=-\infty$, $\lim\limits_{x\to0}\dfrac{1}{x}$ **does not exist.**	

§4. Summary of (ϵ,δ)–Definition Types of Goals and Answers

Here are all 9 possible types of limits (3 possible goals, and 3 possible answers) for $\lim\limits_{x\to\square_1}f(x)=\square_2$:

limit $\square_2$	goal for $f(x)$	$x\to\square_1$	answer for x
L	ϵ-goal around L	$x\to a$	δ-interval around a
$+\infty$	M-goal, $M>0$	$x\to+\infty$	N-answer, $N>0$
$-\infty$	M-goal, $M<0$	$x\to-\infty$	N-answer, $N<0$