Topics for Review for Midterm II in Calculus 1A

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1. Definitions

Be able to **write** precise definitions for any of the following concepts (where appropriate: both in words and in symbols), to **give** examples of each definition, to **prove** that these definitions are satisfied in specific examples. Wherever appropriate, be able to **graph** examples for each definition. What is

(1) the *average velocity* and the *instantaneous velocity* given a displacement function $f(t)$?
(2) the notation $\dfrac{d}{dx}$?
(3) the number e? the properties of e? e^x? What is $\ln x$?
(4) the *composition* $f \circ g$ of two functions $f(x)$ and $g(x)$?
(5) the *inverse* function $f^{-1}(x)$ of a function $f(x)$? How can we obtain their graphs from each other? What is the relationship between their derivatives?
(6) the *second derivative* of $f(x)$?
(7) the *acceleration* and the *jerk* of a moving body?
(8) *implicit* differentiation?
(9) *logarithmic* differentiation? *exponential* differentiation?
(10) *exponential growth* and *decay*?
(11) *continuously compounded interest*?
(12) *linear approximations*? *differential* of a function?
(13) a *global minimum* and a *global maximum* of a function?
(14) a *local minimum* and a *local maximum* of a function?
(15) a *global extremum* and a *local extremum* of a function? Is a local extremum necessarily a global extremum? examples? Is a global extremum necessarily a local extremum? examples? Can an endpoint be a local extremum? a global extremum?
(16) a *critical point*? How do we find all of them? What is a *potential* and a *realized* local extremum?
(17) a *nice* function? a *very nice* function?
(18) a *concave up* (*concave down*) function?
(19) an *inflection* point? How do we locate all of them?
(20) an *indeterminate* form? How do we deal with them?
(21) the method of *contradiction*? How do we use it?
(22) a *slant asymptote*? How do we find it?

2. Theorems

Be able to **write** what each of the following theorems (laws, propositions, corollaries, etc.) says. Be sure to understand, distinguish and **state** the conditions (hypothesis) of each theorem and its conclusion. Be prepared to **give** examples for each theorem, and most importantly, to **apply** each theorem appropriately in problems. The latter means: decide which theorem to use, check (in writing!) that all conditions of your theorem are satisfied in the problem in question, and then state (in writing!) the conclusion of the theorem using the specifics of your problem.

(1) **Differentiation Laws (DL).**
  (a) *Basis Cases.*
    (i) *Constant Functions:* $(c)' = 0$ for any constant c.
    (ii) *Power Rule:* $(x^a)' = a\,x^{a-1}$ for any number a.
    (iii) *Natural Exponential & Logarithmic Functions:* $(e^x)' = e^x$; $(\ln x)' = \frac{1}{x}$ $(x > 0.)$

(iv) *Exponential & Logarithmic Functions:* $(a^x)' = a^x \ln a$ $(a > 0)$; $(\log_a x)' = \dfrac{1}{x \ln a}$ $(x, a > 0.)$

(v) *Trig. Functions:* $(\sin x)' = \cos x$; $(\cos x)' = -\sin x$; $(\tan x)' = \dfrac{1}{\cos^2 x}$; $(\cot x)' = -\dfrac{1}{\sin^2 x}$.

(vi) *Inverse Trig. Functions:*
$(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$; $(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$; $(\arctan x)' = \dfrac{1}{1+x^2}$; $(\operatorname{arccot} x)' = -\dfrac{1}{1+x^2}$.
Domains and graphs of the trig. and inverse trig. functions and their derivatives?

(b) *Multiplication by a Constant:* If $f(x)$ is a differentiable function, then $(c\, f(x))' = c\, f'(x)$.

(c) *Sum and Difference Rules*: If $f(x)$ and $g(x)$ are differentiable, then their sum and difference are also differentiable functions with derivatives given by: $(f(x) + g(x))' = f'(x) + g'(x)$, and $(f(x) - g(x))' = f'(x) - g'(x)$.

(d) *Product Rule:* If $f(x)$ and $g(x)$ are differentiable, then their product is also differentiable, and $\big(f(x) \cdot g(x)\big)' = f'(x)g(x) + f(x)g'(x)$.

(e) *Quotient Rule:* If $f(x)$ and $g(x)$ are differentiable, and $g(x) \neq 0$ for all x nearby a (or on a given interval (A, B)), then their quotient is also differentiable whose derivative is given by:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

(f) *Chain Rule:* If $F(x) = f(g(x))$ for some differentiable functions $f(x)$ and $g(x)$, then $F(x)$ is also differentiable and its derivative is given by: $F'(x) = f'(g(x)) \cdot g'(x)$.

(g) *Some Miscellaneous DLs.* You must understand what they mean how to and use them in problems. You must either be able to derive them on the exam or memorize them.

- *Power and Chain Rules combined:* $(f^n(x))' = n\, f^{n-1}(x)\, f'(x)$; $(f^c(x))' = c\,(f^{c-1}(x))\, f'(x)$ (c is a constant.)
- *Reciprocal Rule:* $\left(\dfrac{1}{f(x)}\right)' = \dfrac{-f'(x)}{f^2(x)}$.
- *Logarithmic and Chain Rules combined:* $(\ln f(x))' = \dfrac{f'(x)}{f(x)}$.
- *Generalized Power Differentiation:* $(c^{f(x)})' = c^{f(x)} \cdot \ln c \cdot f'(x)$.

(2) **Limit Theorems.** $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$; $\lim\limits_{x\to 0}(1+x)^{1/x} = e$.

(3) **Extreme Value Theorem.** To what functions is it applicable?

(4) **Theorem for Local Extrema.** If $f(x_0)$ is a local min/max and $f'(x_0)$ exists, then $f'(x_0) = 0$.

(5) **Contrapositive Statement.** If $f'(x_0) \neq 0$, then $f(x)$ cannot have a local extremum at x_0.

(6) **Converse Statement is False.** If $f'(x_0) = 0$, this does not guarantee that $f(x)$ has a local extremum at x_0. Why? Counterexample?

(7) **Rolle's Theorem.** To what functions is it applicable?

(8) **Mean Value Theorem.** To what functions is it applicable?

(9) **Cor.1.** If $f'(x) = 0$ $\forall x \in (a, b)$, then $f(x)$ is a constant function on (a, b).

(10) **Cor.2.** If $f'(x) = g'(x)$ $\forall x \in (a, b)$, then $f(x)$ and $g(x)$ differ by a constant: $f(x) = g(x) + c$ everywhere on (a, b).

(11) **Monotone Test.** Does it always work? Examples.

(12) **First Derivative Test.** Does it always work? Examples.

(13) **Second Derivative Test.** Does it always work? Examples.

(14) **Concavity Test.** Does it always work? Examples.

(15) **Inflection Point Test.** Does it always work? Examples.

(16) **L'Hospital's Rule.** Does it always work? Examples.

3. Problem Solving Techniques

(1) **How do we find derivatives using DLs?** If you are given $f(x)$ via one formula and you are not asked to use the definition of derivative, you apply DLs. However, for the purposes of this Midterm, the only DLs allowed are listed previously: DLs for polynomials, power rule, DLs for addition, difference and multiplication by a constant.

(a) First see if you can further simplify the given formula. In particular, try to avoid applying the Quotient Rule whenever possible because it is prone to errors. In practice this mean: try to get rid of denominators by either splitting fractions and then simplifying each fraction separately (see formulas for fraction manipulations below), or by direct cancellation of common stuff in the numerator and denominator, or by moving the denominator into the numerator: e.g., x^3 in the denominator becomes x^{-3} in the numerator.

(b) If you are going to apply the Power Rule, turn all expressions like $\sqrt[n]{x}^m$ into the standard form $x^{\frac{m}{n}}$. Again, such expressions in the denominator should move into the numerator wherever suitable by flipping the sign of the power: $\sqrt[n]{x}^m$ in the denominator becomes $x^{-\frac{m}{n}}$ in the numerator.

(c) Look at your function $f(x)$ to figure out its components, the simpler pieces it is made of, and decide which DL(s) you are going to use. In some cases, you may have to apply several DLs one after the other, so keep good track of your intermediate results, or else your calculations will be untraceable. A good strategy is to name some of the simpler components of $f(x)$, e.g., $g(x)$, $h(x)$, etc. and perform some of the necessary differentiation on these functions on the side and then put back your results together. To reduce errors and to make clear that you do know the DLs, it is always good to write the DL formula in terms of functions at first, e.g.,

$$((5x+2)\cdot x^3)' \overset{PR}{=} (5x+2)'\cdot x^3 + (5x+2)\cdot (x^3)'.$$

(d) In case your function is given by several formulas on different intervals, you must find the derivative of each such formula on the corresponding interval. In the end, you must compare your results for the left-side and right-side derivative at the "break" points to determine if you function is differentiable there. e.g., if $f(x)$ is defined by two different formulas on $(2,5]\cup(5,8)$, then at the end you must compare $f'_-(5) \overset{?}{=} f'_+(5)$. If yes, then $f'(5)$ also exists; if not, then $f'(5)$ doesn't exist. You final answer for $f'(x)$ is again going to be given by several different formulas on the corresponding intervals.

(2) **$F(x)$ is a composition of several functions. How do we find these functions?** Start with the variable x wherever it appears in the formula for $F(x)$. See what operations are performed on x and record the corresponding functions from right to left. e.g., $F(x) = \sqrt[3]{2\sin x + 5}$:

$$x \overset{f}{\mapsto} \sin x \overset{g}{\mapsto} 2\sin x + 5 \overset{h}{\mapsto} \sqrt[3]{2\sin x + 5}.$$

Name the intermediate expressions by some variables, say, $u = \sin x$, $v = 2\sin x + 5 = 2u + 5$. Thus,

$$x \overset{f}{\mapsto} u = \sin x \overset{g}{\mapsto} v = 2u + 5 \overset{h}{\mapsto} \sqrt[3]{v}.$$

From here we see that $f(x) = \sin x$, $g(u) = 2u + 5$ and $h(v) = \sqrt[3]{v}$. Hence, $F(x) = h(g(f(x))) = (h\circ g\circ f)(x)$, a triple composition. The above method is "from inside-out", i.e., start with the inner functions and move to the outer functions.

(3) **How do we apply the Chain Rule to compositions of three or more functions?** The name "Chain Rule" comes from the expressions "chain reaction". It is a recursive process, but to keep things simple see what happens to the example above:

$$F'(x) = h'(g(f(x)))\cdot g'(f(x))\cdot f'(x).$$

We first find, on the side, all involved derivatives:

$$f'(x) = (\sin x)' = \cos x,\ g'(u) = (2u+5)' = 2,\ h'(v) = (\sqrt[3]{v})' = (v^{1/3})' = \frac{1}{3}v^{\frac{1}{3}-1} = \frac{1}{3}v^{-\frac{2}{3}} = \frac{1}{3v^{\frac{2}{3}}}.$$

We then substitute back: $F'(x) = \frac{1}{3v^{\frac{2}{3}}} \cdot 2 \cdot \cos x$. Now we recall our substitutions above for $u = \sin x$ and $v = 2\sin x + 5$:

$$F'(x) = \frac{1}{3(2\sin x+5)^{\frac{2}{3}}} \cdot 2 \cdot \cos x = \frac{2\cos x}{3(2\sin x+5)^{\frac{2}{3}}} = \frac{2\cos x}{3\sqrt[3]{(2\sin x+5)^2}}.$$

One may also try the opposite way: start with the outermost (or last) function and move backwards to the innermost (first) function. So here goes your favorite **blah**-method:

$$(\sqrt[3]{2\sin x+5})' = \left((2\sin x+5)^{\frac{1}{3}}\right)' = \left((\text{blah})^{\frac{1}{3}}\right)' \cdot (\text{blah})' = \frac{1}{3}(\text{blah})^{\frac{1}{3}-1} \cdot (\text{blah})' =$$

$$= \frac{1}{3}(2\sin x+5)^{-\frac{2}{3}} \cdot (2\sin x+5)' = \frac{1}{3}(2\sin x+5)^{-\frac{2}{3}} \cdot (2\cos x) = \frac{2\cos x}{3\sqrt[3]{(2\sin x+5)^2}}.$$

(4) **But how do we apply the Chain Rule if there are several occurrences of x?** Several occurrences of x ordinarily means that one chain rule will not suffice, and you will have to apply some other DLs. For example, if $G(x) = \sqrt[3]{2\sin x+5} \cdot 7^{\tan x}$, then the "biggest" (or last) operation is the multiplication in the middle, so we'll have to apply the product rule to the two functions $F(x) = \sqrt[3]{2\sin x+5}$ and $H(x) = 7^{\tan x}$:

$$G'(x) = \big(F(x) \cdot H(x)\big)' = F'(x)H(x) + F(x)H'(x).$$

In this case, obviously, to keep our sanity, we'd better differentiate $F(x)$ and $H(x)$ on the side (as if they were two other simpler but independent problems), and then substitute our answers back for $G'(x)$. Thus, good record keeping is of utmost importance here.

(5) **But how do we find the derivative of such an exponential function as $H(x) = 7^{\tan x}$?** Represent $H(x)$ as the composition of two functions: $H(x) = f(g(x))$, where $g(x) = \tan x$ and $f(u) = 7^u$, $u = g(x) = \tan x$. The derivatives of these functions are as follows:

$$g'(x) = (\tan x)' = \frac{1}{\cos^2 x},\ f'(u) = (7^u)' = 7^u \cdot \ln 7.$$

If you forget the derivative of the tangent function, then proceed with the quotient rule there. The derivative of 7^u was found by applying the Exponential Functions Rule. Now, the Chain Rule implies:

$$H'(x) = f'(g(x)) \cdot g'(x) = (7^u \cdot \ln 7) \cdot \frac{1}{\cos^2 x} = 7^{\tan x} \cdot \ln 7 \frac{1}{\cos^2 x} = \frac{7^{\tan x} \cdot \ln 7}{\cos^2 x}.$$

Again, the people rooting for the **blah**-method will enjoy the following:

$$\left(7^{\tan x}\right)' = \left(7^{\text{blah}}\right)' \cdot (\text{blah})' = 7^{\text{blah}} \ln 7 \cdot (\text{blah})' = 7^{\tan x} \cdot \ln 7 \cdot (\tan x)' = 7^{\tan x} \cdot \ln 7 \cdot \frac{1}{\cos^2 x} = \frac{7^{\tan x} \cdot \ln 7}{\cos^2 x}.$$

(6) **But still, what will be the derivative of $G(x) = \sqrt[3]{2\sin x+5} \cdot 7^{\tan x}$? Wouldn't that be too complicated?** Yes, the answer would be a heck of an answer if we manage to get to it without making errors, but remember that most of these calculations are algorithmic, i.e., there are rules which can be applied and not too much thinking is required; indeed, you have to be ruthlessly precise and keep your concentration throughout the whole computation process. Besides, who said that life is simple? :) In any case, here it goes: we have found the derivatives of $F(x) = \sqrt[3]{2\sin x+5}$ and $H(x) = 7^{\tan x}$ above. Because we kept good records of what we did, we can now substitute our results into the product rule for $G(x) = F(x) \cdot H(x)$:

$$G'(x) = F'(x)H(x) + F(x)H'(x) = \frac{2\cos x}{3\sqrt[3]{(2\sin x+5)^2}} \cdot 7^{\tan x} + \sqrt[3]{2\sin x+5} \cdot \frac{7^{\tan x} \cdot \ln 7}{\cos^2 x}.$$

(7) **Gosh, what about the derivatives of inverse functions? Do we need to know these too?** A good advise is to have the derivatives of the inverse trigonometric functions on your cheat sheet, AND to understand the graphical ideas behind the derivatives of inverse functions.

If two functions $f(x)$ and $g(y)$ are inverses of each other, then their two compositions must yield the identity function, i.e., $f(g(y)) = y$ and $g(f(x)) = x$. For example, $f(x) = e^x$, and $g(y) = \ln y$.

(a) If we want to draw the graphs of these two functions on the same coordinate system, it will be silly to draw one function in terms of x, and the other in terms of y. (Why? Because then the two graphs will literally coincide, and we won't be able to distinguish between the two functions!! Check out this phenomenon for, say, x^2 and $\sqrt{y}$.) Thus, for graphing purposes, we relabel one of the variables, say, $g(x) = \ln x$, and draw the graphs of $f(x) = e^x$ and $g(x) = \ln x$.

(b) The graphs of two inverse functions are symmetric across the line $y = x$. This means that if $f(x)$ passes through point $(2, 7)$, then $g(x)$ passes through point $(7, 2)$, and conversely. In our particular example above, verify that $f(x)$ passes through points $(0, 1)$ and $(1, e)$, while $g(x)$ passes through points $(1, 0)$ and $(e, 1)$.

(c) The corresponding tangent lines will also be symmetric across the line $y = x$. Thus, the tangent line to the graph of e^x at point $(0, 1)$ will be symmetric to the tangent line to the graph of $\ln x$ at point $(1, 0)$, etc. (Check it on graphically.) This means that the slopes of these two tangent lines will be reciprocal.

(d) To finish off, go back to the original variable $g(y)$, or else there will be a mess. Translating the above into derivatives, we summarize: for any pair of inverse functions $f(x)$ and $g(y)$, their derivatives at the corresponding points are reciprocal to one another:

$$g'(y) = \frac{1}{f'(x)}, \quad \text{where } y = f(x),\ x = g(y).$$

For example, if $y = f(x) = x^2$ and $x = g(y) = \sqrt{y}$ $(x > 0)$, then we can verify that

$$f'(x) = 2x,\ g'(y) = (\sqrt{y})' = \frac{1}{2\sqrt{y}},$$

$$g'(y) \stackrel{?}{=} \frac{1}{f'(x)} \quad \Leftrightarrow \quad \frac{1}{2\sqrt{y}} \stackrel{?}{=} \frac{1}{2x} \quad \Leftrightarrow \quad \frac{1}{2\sqrt{x^2}} \stackrel{?}{=} \frac{1}{2x} \quad \Leftrightarrow \quad \frac{1}{2x} \stackrel{?}{=} \frac{1}{2x} \quad \text{Yes.}$$

In particular, for the point $(3, 9)$ on the graph of $f(x) = x^2$: $f'(3) = 6$ while $g'(9) = 1/6$. (We took $g'(9)$, *not* $g'(3)$, since the point $(9, 3)$ lies on the graph of g.) So, indeed, the slopes $f'(3)$ and $g'(9)$ are reciprocal numbers.

(e) **(Extra Stuff. Suitable for Bonus Questions)**. Let's find the derivative of, say, $\operatorname{arccot} x$. We set the pair of inverse functions $y = f(x) = \cot x$, and $x = g(y) = \operatorname{arccot} y$. According to our formula above:

$$g'(y) = \frac{1}{f'(x)}, \text{ i.e., } (\operatorname{arccot} y)' = \frac{1}{(\cot x)'} = \frac{1}{-\frac{1}{\sin^2 x}} = -\sin^2 x.$$

But this is in the **wrong** variable x! And just replacing x by y wouldn't yield the right thing because we have defined x above by $x = \operatorname{arccot} y$. We could plug this in and obtain: $g'(y) = -\sin^2(\operatorname{arccot} y)$, but this is such a horribly inconvenient and uninsightful formula, that we might as well leave the original $g'(y) = (\operatorname{arccot} y)'$.

The solution to this dilemma is as follows, and this is the tricky part. Rewrite somehow $\sin^2 x$ in terms of the original function $\cot x$, and then we'll follow our noses and substitute. OK, but how can we do it? Recall that $\cot x = \frac{\cos x}{\sin x}$, so squaring both sides gives

$$\cot^2 x = \frac{\cos^2 x}{\sin^2 x} = \frac{1 - \sin^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} - 1$$

Solving now for $\sin^2 x$ yields $\sin^2 x = \dfrac{1}{1+\cot^2 x}$. This is it! Now we have

$$(\operatorname{arccot} y)' = -\sin^2 x = -\frac{1}{1+\cot^2 x} = -\frac{1}{1+y^2}.$$

Thus, $(\operatorname{arccot} y)' = -\frac{1}{1+y^2}$, or written in the more traditional notation (now we can relabel y as x if we wish): $(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$.

(8) **How do we use implicit differentiation?** This is used to find tangent lines and their slopes to *curves* in the plane which are *not* graphs of functions (i.e., they violate the vertical line test). Thus, we don't have a function formula to differentiate, but instead an equation for the curve, e.g., $x^3 + x^2y + 4y^2 = 6$. It will be hard, sometimes impossible, to solve such an equation for y, and hence a formula for y may not be available. [1]

(a) We imagine that y is given by such a formula $y = f(x)$ (e.g., $x^3 + x^2 f(x) + 4f(x)^2 = 6$), and we differentiate (with respect to x) both sides of the given equation, e.g.,

$$(x^3 + x^2y + 4y^2)' = (6)' \;\Rightarrow\; 3x^2 + 2xy + x^2y' + 8y \cdot y' = 0$$

Do *not* forget to include y' wherever appropriate: $y = f(x)$ so that $y' \neq 1$, but $y' = \frac{dy}{dx} = f'(x)$.

(b) Solve the above for y':

$$3x^2 + 2xy + y'(x^2 + 8y) = 0 \;\Rightarrow\; y' = -\frac{3x^2 + 2xy}{x^2 + 8y}.$$

This is the best we can do for y': we have expressed it in terms of x and the original function y.

(c) If we are asked something about derivatives, slopes and tangents at specific places, then we use the above formula for y' and if necessary, the original equation for y. e.g., in our example, find the slope and the equation for the tangent at point $(1, 1)$. [2] Now we use the formula for $y'(x)$ and substitute $x = 1$, $y = 1$:

$$y'(1) = -\frac{3+2}{1+8} = -\frac{5}{9}.$$

Finally, we use the point-slope formula:

$$y'(1) = \frac{y - y(1)}{x - 1} \;\Rightarrow\; -\frac{5}{9} = \frac{y-1}{x-1} \;\Rightarrow\; y = -\frac{5}{9}x + \frac{14}{9}.$$

It is always good to check if this is the correct equation for the tangent line: yes, because the slope is $-5/9$, and if we plug in the point $(1, 1)$ it works: $1 = 1$.

(d) Say, we want to find all points on the curve where the tangent to the curve is horizontal. In general, this is not an easy question to answer. Set $y'(x) = 0$, and obtain two equations in terms of x and y: the derivative equation and the original equation. Now you are supposed to solve this system of two equations for x and y. In our example, this amounts to:

$$\left|\begin{array}{l} 0 = -\frac{3x^2+2xy}{x^2+8y} \\ x^3 + x^2y + 4y^2 = 6 \end{array}\right.$$

The first equation yields $0 = 3x^2 + 2xy = x(3x + 2y)$, i.e., $x = 0$ or $y = -\frac{3}{2}x$. Substituting in the second equation: $4y^2 = 6$ (when $x = 0$), or $x^3 - \frac{3}{2}x^3 + 4\frac{9}{4}x^2 = 6$ (when $y = -\frac{3}{2}x$), i.e., $y = \pm\sqrt{\frac{3}{2}}$, while the second equation is a pain and I won't solve it here. The final answer would have been: the tangent lines to the curve are horizontal at points $(0, \sqrt{\frac{3}{2}})$, $(0, -\sqrt{\frac{3}{2}})$, and at the points yielded by the second case above.

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[1] In the particular example, you can indeed solve for y viewing the given equation as a quadratic equation in "y", but believe me, you probably don't want to do that, and you should follow the method of implicit differentiation instead.

[2] Note that this point was obtained by substituting $x = 1$ into the original equation: $1 + y + 4y^2 = 6$ and obtaining solutions $y = 1, -5/4$. Thus, you could have been asked to find instead the tangent line at point $(1, -5/4)$.

The good news is that if a similar question appears on the exam, the calculations will be easier. The method, however, is outlined above. Note that similarly you can solve all sorts of questions about the tangents to such curves: e.g., find where the tangents are parallel to $y = x$ (set $y'(x) = 1$), etc.

(9) **How do we use Logarithmic Differentiation?** Usually, this is used to find the derivatives of functions of the form: $F(x) = f(x)^{g(x)}$. The Chain Rule here doesn't apply easily (why?), so instead we "ln" both sides of the given equation to get rid of the exponent:

$$\ln F(x) = \ln f(x)^{g(x)} = g(x) \ln f(x).$$

In the last step we used the identity $\ln a^b = b \ln a$. Differentiate both sides of the obtained equality: $(\ln F(x))' = (g(x) \ln f(x))'$. On the LHS we use the chain rule, and on the RHS we use the product rule and then the chain rule:

$$\frac{F'(x)}{F(x)} = g'(x) \ln f(x) + g(x)(\ln f(x))'$$

$$\frac{F'(x)}{F(x)} = g'(x) \ln f(x) + g(x) \frac{f'(x)}{f(x)}.$$

Now we recall that we really wanted to get $F'(x)$, so we solve for it:

$$F'(x) = F(x) \cdot \left(g'(x) \ln f(x) + g(x) \frac{f'(x)}{f(x)}\right) = f(x)^{g(x)} \left(g'(x) \ln f(x) + g(x) \frac{f'(x)}{f(x)}\right).$$

The resulting formula is so complicated, that I do not advise you to remember it or even to write it on your cheat sheet. Apply the above procedure to a specific example and work out the whole thing from scratch: it will be easier than to try to apply the above formula.

(10) **How do we use Exponential Differentiation?** As logarithmic differentiation, exponential differentiation can be used to find the derivatives of functions of the form: $F(x) = f(x)^{g(x)}$. This time we represent the function $F(x)$ as an exponent using the formula $e^{\ln y} = y$ where $y = F(x)$:

$$F(x) = e^{\ln F(x)} = e^{\ln f(x)^{g(x)}} = e^{g(x) \ln f(x)}.$$

Now we are in good shape since we can apply the Chain Rule:

$$F'(x) = e^{g(x) \ln f(x)} \cdot (g(x) \ln f(x))'.$$

No wonder that this yields exactly the same (complicated) formula as logarithmic differentiation above. Thus, you have to choose for yourselves which of the two methods you prefer: logarithmic or exponential differentiation, and learn to apply the chosen method on the spot to specific examples.

(11) **What's the use of the Limit Theorem $\lim_{x \to 0} \frac{\sin x}{x} = 1$ other than applying it to prove that $(\sin x)' = \cos x$?** We already saw how the knowledge of this limit helped us show that

$$\lim_{x \to 0} \frac{\cos x - 1}{x} = 0.$$

One can find many more limits now (to which limit laws are of no use.)

(a) The first idea is to complete, wherever possible, expressions to look like the limit above. e.g.,

$$\lim_{x \to 0} \frac{\sin 5x}{3x} = \lim_{x \to 0} \frac{\sin 5x}{5x} \cdot \frac{5}{3} = \frac{5}{3} \lim_{x \to 0} \frac{\sin 5x}{5x}.$$

Now we can substitute $u = 5x$, and note that when $x \to 0$ then $u \to 0$:

$$\frac{5}{3} \lim_{u \to 0} \frac{\sin u}{u} = \frac{5}{3} \cdot 1 = \frac{5}{3}.$$

(b) The second idea is to "force" expressions of the type $(\sin x)/x$ by rewriting our functions, to isolate the "trouble-makers"; to apply to them the above limit theorem, and to apply to the rest the limit laws. e.g., in the following example, LL fail when $x = 0$, so we rewrite the function:

$$\lim_{x\to 0} \frac{\sin x}{x + \tan x} = \lim_{x\to 0} \frac{\sin x}{x + \frac{\sin x}{\cos x}} = \lim_{x\to 0} \frac{\sin x \cos x}{x \cos x + \sin x}.$$

Now, we see that $\cos x$ is no trouble at all since for $x = 0$ it gives $\cos 0 = 1$ and this doesn't mess anything. We can easily isolate the top $\cos x$ by LL for product, but we can't isolate the bottom one so easily. However, if we were to apply the Limit Theorem, each of the two $\sin x$ would require an x "underneath". We achieve this by factoring x in the denominator:

$$\lim_{x\to 0} \cos x \cdot \lim_{x\to 0} \frac{\sin x}{x(\cos x + \frac{\sin x}{x})} = \cos 0 \cdot \frac{\lim_{x\to 0} \frac{\sin x}{x}}{\cos 0 + \lim_{x\to 0} \frac{\sin x}{x}} = 1 \cdot \frac{1}{1+1} = \frac{1}{2}.$$

(12) **How do we apply Rolle's Thm in problems?** Rolle's Thm is used typically in problems of the type: *Show that a very nice function $f(x)$ has at most k roots.*

Argue by contradiction. Start by supposing that $f(x)$ has more than k roots, i.e., that $f(x)$ has at least $k+1$ roots. Then between every two consecutive roots, by Rolle's Thm, the derivative function $f'(x)$ must have at least one root. Thus, $k+1$ roots of $f(x)$ mean that $f'(x)$ has at least k roots on its own. Study $f'(x)$ and show that it does **not** have these many roots. (For this, you may need to go down to $f''(x)$, or even higher derivatives, if necessary; but usually, a check of $f'(x)$ is sufficient.)

So, on the one hand, $f'(x)$ must have k roots, but on the other hand, it doesn't have k roots - well, that's a contradiction. Conclude that the supposition was wrong to start with. Finally, conclude that therefore $f(x)$ does indeed have at most k roots. $\square$

(13) **How do we apply MVT and its Corollaries in problems?**

(a) *Show that some identity is true.* First, label the two sides of the identity by $f(x)$ and $g(x)$. Find the two derivatives $f'(x)$ and $g'(x)$ and show that they are equal everywhere: $f'(x) = g'(x)\ \forall x$. Conclude by Cor.2 that $f(x) = g(x) + c$ for some constant c. Finally, substitute your favorite $x = a$ and show that $f(a) = g(a)$ (well, make sure that you choose your a so that you can easily calculate $f(a)$ and $g(a)$.) But from the above, $f(a) = g(a) + c$, so that $c = 0$. Conclude that $f(x) = g(x)$ everywhere. $\square$

(b) *Given some data about a function $f(x)$, show some inequality involving $f(x)$ and or $f'(x)$.* There are various problems of this type, and we can't describe in detail all of them. The general principle is as follows. Set up the MVT for $f(x)$ on convenient interval $[A, B]$:

$$f'(x_0) = \frac{f(B) - f(A)}{B - A} \text{ for some } x_0 \in (A, B).$$

Plug in whatever data you have about $f(x)$ or about the derivative $f'(x)$ and see what this implies. Usually, this quickly leads to whatever you are asked to show. $\square$

(14) **How do we find where $f(x)$ is increasing or decreasing?**

(a) Use the *Increasing/Decreasing Test.* If $f'(x) > 0$ everywhere on some interval, then $f(x)$ increases on this interval. If $f'(x) < 0$ everywhere on some interval, then $f(x)$ decreases on this interval. If $f'(x) = 0$ everywhere on some interval, then $f(x)$ is constant on this interval (by Cor.1.) Thus, find for which x's $f'(x) > 0$, for which $f'(x) < 0$.

(b) Be careful when $f'(x_0) = 0$: just the fact that $f'(x_0) = 0$ does **not** mean that the function is constant nearby x! (e.g., x^3 nearby $x = 0$.) Thus, check the sign of the derivative at nearby points and this will tell you what is going on exactly at $f'(x_0)$. For example, if $f'(x) > 0$ both on the left and right of x_0, then the function is increasing at x_0 too. But if $f'(x)$ changes sign at x_0, then $f(x_0)$ is a local extremum so that $f(x)$ is neither increasing nor decreasing at x_0. $\square$

(15) **How do we find the global extrema of $f(x)$?**

(a) First, find where $f'(x_0) = 0$ and record all such $f(x_0)$.

(b) Next, record all $f(x_0)$ where derivative does **not** exist, and also check and record what happens at the ends of the intervals (if there are any ends, that is.)

(c) Compare all values $f(x_0)$ at the critical points recorded above and find which are your global min/max. Be careful if the function is defined on an infinite interval or has vertical asymptotes. This gives opportunity for $f(x)$ to have arbitrarily large (small) values: some limit(s) of $f(x)$ could be $\pm\infty$, so in such cases, global min/max may not exist. $\square$

(16) **How do we find the local extrema of** $f(x)$**?** For simplicity of exposition, let's assume that $f(x)$ is a very nice function, so we don't have to worry for now about discontinuities or non-existing derivatives.

(a) First find the potential local extrema of $f(x)$, i.e., check where $f'(x_0) = 0$ and record all such places x_0.

(b) Next, use the *Second Derivative Test* (if $f''(x_0)$ exists, of course) to determine which of these potential extrema are realized local extrema. In particular, if $f''(x_0) > 0$ then $f(x_0)$ is a local minimum; if $f''(x_0) < 0$ then $f(x_0)$ is a local maximum.

(c) If $f''(x_0) = 0$ or the second derivative $f''(x_0)$ does not exists (the latter could happen even if $f(x)$ is a very nice function!), then the Second Derivative Test fails to give us anything useful, so we try the *First Derivative Test.* If $f'(x)$ changes its sign at x_0 from $+$ to $-$, then $f(x_0)$ is a local maximum; if $f'(x)$ changes its sign at x_0 from $-$ to $+$, then $f(x_0)$ is a local minimum. If $f'(x)$ does **not** change its sign at x_0, then $f(x_0)$ is **not** a local extremum. $\square$

(d) The *First Derivative Test* is **bulletproof!!!**

(17) **How do we find where the function is concave up or down?**

(a) Use the *Concavity Test.* If $f''(x) > 0$ everywhere on some interval, then $f(x)$ is concave-up on this interval. If $f''(x) < 0$ everywhere on some interval, then $f(x)$ is concave-down on this interval.

(b) If $f''(x_0) = 0$, then there could be an *inflection point* at x_0, so we check if $f''(x)$ changes sign at x_0. If yes, then at x_0 the function does indeed have an inflection point: the function changes from concave-up to concave-down or the other way around. But if $f''(x)$ does **not** change sign at x_0, then the function does **not** have an inflection point at x_0; and $f(x)$ continues to be concave-up (or concave-down) as it were before.

(c) As an example, compare the two functions x^3 and x^4 nearby $x_0 = 0$ and determine if they have inflection points at 0. Further, determine the intervals where these functions are concave-up or concave-down. $\square$

(18) **How do we use L'Hospital's Rule?** L'Hospital's Rule is applied to find limits **only** in cases where Limit Laws fail! Do **not** apply L'Hospital's Rule (LH) in cases where LLs work, or you will get an **incorrect** result. In other words, LH and LLs are *complements of each other*: one works exactly where the other doesn't work.

(a) The cases where LH applies can be determined as follows. Attempt to apply LLs and get an *indeterminate* of one of the types:

$$\frac{0}{0},\ \frac{\pm\infty}{\pm\infty},\ 0\cdot(\pm\infty),\ 0^0, \infty^0, 1^\infty.$$

None of these expressions make sense (and they cannot be given any reasonable uniform sense no matter how hard we try!)

(b) LH applies *directly* only to the **quotient** indeterminants: $0/0$ and $\pm\infty/\pm\infty$. Thus, if we deal with a limit like:

$$\lim_{x\to a}\frac{f(x)}{g(x)} \overset{LL}{=} \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \overset{LL}{=} \frac{0}{0} \text{ or } \frac{\pm\infty}{\pm\infty},$$

we can replace each of $f(x)$ and $g(x)$ by their derivatives without changing the overall limit:

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}. \quad \square$$

(c) Any of the other indeterminate forms $0\cdot\infty,\ 0^0,\infty^0,1^\infty$ must be rewritten to fit one of the above two basic cases before we can even start applying LH.

- **Product** Indeterminacy $0\cdot\infty$ means that we have a limit like: $\lim_{x\to a} f(x)\cdot g(x)\overset{LL}{=} 0\cdot\infty$. We rewrite this simply by taking the reciprocal of $f(x)$ or $g(x)$, whichever is more convenient:

$$\lim_{x\to a} f(x)\cdot g(x)=\lim_{x\to a}\frac{g(x)}{\frac{1}{f(x)}}\overset{LL}{=}\frac{\pm\infty}{\pm\infty},\ \text{or}\ \lim_{x\to a} f(x)\cdot g(x)=\lim_{x\to a}\frac{f(x)}{\frac{1}{g(x)}}\overset{LL}{=}\frac{0}{0}.$$

Thus, we apply LH to the new rewritten expressions above. $\square$

- **Exponential** Indeterminacies $0^0,\infty^0,1^\infty$ mean that we have a limit like

$$\lim_{x\to a} f(x)^{g(x)}\overset{LL}{=}0^0,\infty^0 \text{ or } 1^\infty.$$

This is a tougher case to handle. Recall the basic identity

$$e^{\ln blah}=blah,\ \text{or equivalently}\ blah=e^{\ln blah}.$$

Setting $blah=f(x)^{g(x)}$, we rewrite:

$$f(x)^{g(x)}=e^{\ln\left(f(x)^{g(x)}\right)}=e^{g(x)\cdot\ln f(x)}.$$

Now instead of trying to find the limit of the whole expression, we can concentrate on finding the limit of the exponent only: $\lim_{x\to a} g(x)\cdot\ln f(x)$. If we apply LLs here, we will necessarily get an indeterminacy form $0\cdot\pm\infty$ or $\pm\infty\cdot 0$, with which we know how to deal from the previous discussion.

To summarize, to find $\lim_{x\to a} f(x)^{g(x)}$, where LLs would produce an indeterminacy of the form $0^0,\infty^0$ or 1^∞, we first rewrite the expression via an exponential as above. Then we find the limit $\lim_{x\to a} g(x)\cdot\ln f(x)$. Say, this limit turns out to be some number L. Therefore, the original limit will be e^L. $\square$

(d) **Word of caution.** The idea of LH is to *simplify* our job of finding limits, **not** to complicate it! Thus, before you apply LH, find if LLs apply! Do not skip this step - this is not just an advice, this is a necessity! Only after you verify that LLs do **NOT** apply, you can start thinking of applying LH. And even then, it could be that LLs can get part of the job done - so use LLs as much as you can, and leave for LH to take care of the rest.

Why not just use LH on everything? Because LH may not apply; and even if it applies, if not used wisely it will yield a more complicated expression with which you will not know what to do! Remember: derivatives can be simpler than the original functions (e.g., polynomials), but sometimes derivatives can be much more complex than the original functions (e.g., often after you apply the Product Rule, Quotient Rule or Chain Rule).

So, LH is a powerful weapon, but only when used wisely! It can really backfire if we don't know how to use it!

(19) **How do we find slant asymptotes of $f(x)$?**

(a) Find the limits $\lim\limits_{x\to\pm\infty}\dfrac{f(x)}{x}$. If it turns out that this limit does not exists, or exists but is not a finite number, then there will be no slant asymptote in the corresponding direction $x\to\infty$ or $x\to-\infty$. If this limit exists and is a finite number, then call it m:

$$m=\lim_{x\to\infty}\frac{f(x)}{x}.$$

m will be the slope of the slant asymptote.

(b) After we find the slope m, we proceed to find the constant b which will appear in the equation of the slant asymptote. To do that, find the limit

$$b = \lim_{x\to\infty} (f(x) - mx).$$

Thus, the slant asymptote in the direction $x \to \infty$ will be $y = mx + b$. $\square$

Note that there is a short-cut for finding slant asymptotes in case we are looking at a fraction of two polynomials: $f(x)/g(x)$. For clarity, let $f(x) = Ax^m +$ lower degree terms, and $g(x) = Bx^n +$ lower degree terms; in other words, the leading coefficients of f and g are A and B, and their degrees are m and n respectively.

(a) If $m = n$ (i.e., $f(x)$ and $g(x)$ have the same degrees), then

$$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \frac{A}{B} \quad \text{(Why?)}$$

and hence $f(x)/g(x)$ will have a horizontal asymptote $y = A/B$ in *both* directions $x \to \pm\infty$.

(b) If $m < n$ (i.e., the degree of $f(x)$ is less than the degree of $g(x)$), then

$$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = 0 \quad \text{(Why?)}$$

and hence $f(x)/g(x)$ will have a horizontal asymptote $y = 0$ in *both* directions $x \to \pm\infty$.

(c) If $m = n + 1$ (i.e., the degree of $f(x)$ is exactly 1 more than the degree of $g(x)$), then there will be a *slant* asymptote in both directions (why?) More precisely, to find this slant asymptote, we divide by x:

$$\lim_{x\to\pm\infty} \frac{f(x)}{xg(x)} = \frac{A}{B} \quad \text{(Why?)}$$

Hence the slant asymptote will have slope A/B in *both* directions $x \to \pm\infty$. To find the term b from linear equation of the slant asymptote, we proceed as follows:

$$b = \lim_{x\to\pm\infty} \frac{f(x)}{g(x)} - \frac{A}{B}x = \lim_{x\to\pm\infty} \frac{Bf(x) - Ax}{Bg(x)}.$$

There is a formula for this in terms of the second coefficients of $f(x)$ and $g(x)$, but it is not worth remembering. Instead, dutifully find the above limit and this will give your b in the slant asymptote formula.

(d) If $m > n + 1$ (i.e., degree of $f(x)$ is 2 or more than the degree of $g(x)$), then there will be no horizontal or slant asymptotes (in any directions). The reason is that in this case:

$$\lim_{x\to\pm\infty} \frac{f(x)}{xg(x)} = \pm\infty$$

depending on whether the degrees of $f(x)$ and $g(x)$ are odd, even, etc. (Why?)

(e) In conclusion, if you are asked about horizontal and slant asymptotes for a fraction of two polynomials, you can use the above discussion. In particular, the only case where there will be a slant asymptote is when the degree of $f(x)$ is exactly 1 more than the degree of $g(x)$.

(20) **How do we sketch graphs of functions?** It will be very long to describe the whole process for an arbitrary function $f(x)$. On the exam, you will be given a specific function $f(x)$ to which some of the stuff below will apply, and some stuff will possibly be irrelevant.

(a) Establish where $f(x)$ is defined, where it is continuous and where it is differentiable.

(b) Establish if the function has any period. This is relevant usually in cases involving only trigonometric functions. If you find such a period, say, 2π, concentrate below only on one such interval, say $[0, 2\pi]$.

(c) Check if the function is odd or even - this will further reduce your work to concentrating only on, say, the interval $[0, +\infty)$. Such cases may occur with polynomials, fractions of polynomials, adn trig. functions. This step is not strictly necessary, but sometimes it is good to take it into account for more precise sketches.

(d) Find $f'(x)$ and $f''(x)$. Simplify them. If you get a fraction of polynomials, try to cancel everything in common and to factor top and bottom polynomials as much as possible.

(e) Establish where $f(x)$ is increasing and where it is decreasing, using $f'(x)$ and the relevant test(s).

(f) Find the local and global extrema (if any) of $f(x)$, using $f'(x)$, $f''(x)$ and the relevant test(s).

(g) Find where the function is concave-up and concave-down, and all inflection points, using $f''(x)$ and the relevant test(s).

(h) Find the vertical asymptotes (if any). They will appear where $f(x)$ has an infinite (at least) one-sided limit, i.e., if $\lim\limits_{x\to a^+} f(x) = \pm\infty$ or $\lim\limits_{x\to a^-} f(x) = \pm\infty$; then the vertical line $x = a$ is such an asymptote. If $f(x)$ is a fraction, such solutions will be produced by the roots of the denominator.

(i) Find the horizontal asymptotes: these will appear where $f(x)$ has a finite limit when $x \to \pm\infty$, i.e., if $\lim\limits_{x\to\pm\infty} f(x) = L$; then the horizontal line $y = L$ is such an asymptote. If $f(x)$ is a fraction, both numerator and denominator will be involved in this step. Note that L'Hospital's Rule may or may not be needed here.

(j) Find the slant asymptotes (if any). Note that L'Hospital's Rule may or may not be needed here.

(k) If possible, find the roots of the function, i.e., its x-intercepts: try to solve $f(x) = 0$ if possible. If $f(x)$ is a fraction, such solutions will be produced in the numerator. (The denominator will be irrelevant in this step.) It is also good to find the y-intercept, by setting $x = 0$ in $f(x)$.

(l) The above information should be enough to sketch a very good graph of $f(x)$.
- On the x-axis, mark all interesting points which you found above, mark the intervals where $f(x)$ is increasing or decreasing (or constant, for that matter). Mark the x-intercepts and the y-intercept (if any). Mark any local and global extrema you found. Mark any inflection points you found.
- Draw the vertical, horizontal and slant asymptotes (if any). Make a mental picture of what is happening nearby each of these asymptotes. Be careful nearby the vertical asymptotes to reflect whether a given one-sided limit is $+\infty$ or $-\infty$.
- At this moment, if you are not sure how the graph looks in some intervals, plot a few other points there.
- All that is left is to connect the points you plotted respecting the properties you found above: domain of definition, increase/decrease, extrema, concavity and inflection points, dis-continuity and non-differentiability, asymptotes, period of function, odd/even function. □

(21) **How do we cook up a function $g(x)$ which captures the sign of $f'(x)$ or of $f''(x)$?**

Speaking of shortcuts, this may be the single most significant shortcut in drawing graphs: when applicable, this shortcut may allow you to establish all intervals of increase, decrease, concavity, loc. min's and max's **without** having to plug in a single value into $f'(x)$ and $f''(x)$! Now, that's what **I** call a shortcut!

An example illustrates the technique. Say, $f'(x)$ equals the following stupendous expression:

$$f'(x) = \frac{4(x-1)^3(x+7)^6(x-4)^{13}(x^2+10)}{(x+2)^5(x-2)^2(2x^2+3)^2(x+8)}.$$

Naturally, we are interested where this expression is positive, negative and zero. Now, you can plug in values into $f'(x)$ and fill in a table with $+$ and $-$ signs until you are blue in the face, but ... let's do it the **cool way**!

(a) First of all, find where $f'(x) = 0$. This comes from the numerator: $x = 1,\ -7,\ 4$. These are the **potential** local min's and max's of the original function $f(x)$. Note that the factor $x^2 + 10$ does not contribute any zeros (why?)

(b) Let's not forget where $f'(x)$ is **not** defined, just so that we don't write meaningless things later on: this comes from the denominator, i.e., $x \neq -2, 2, -8$. Note that $2x^2 + 3$ does not mess the domain of $f'(x)$ (why?)

(c) Now, we throw out all **irrelevant** terms in $f'(x)$, about which we already know that they are **always** positive. In our case, $(x + 7)^6$, $x^2 + 10$, $(x - 2)^2$, $(2x^2 + 3)^2$ and 4 can all go, since they are always positive. We can't quite throw out $(x - 1)^3$ because it can be either positive or negative, but we can throw out a big chunk of it: $(x - 1)^3 = (x - 1)^2 \cdot (x - 1)$, so $(x - 1)^2$ can happily go, leaving only $(x - 1)$. In a similar fashion, let's get rid of $(x - 4)^{12}$, leaving $(x - 4)$ in the numerator.
In the denominator, we have $(x + 2)^5$. Instead of throwing out $(x + 2)^4$ and leaving $(x + 2)$ in the denominator, it is better to multiply top and bottom by an extra $(x + 2)$:

$$\frac{1}{(x+2)^5} = \frac{(x+2)}{(x+2)^6}.$$

Now, we happily throw out $(x+2)^6$, and we are left with $(x+2)$ in the **numerator**! In a similar vein, multiply top and bottom by $(x + 8)$, throw out $(x + 8)^2$ from the denominator, leaving $(x + 8)$ in the numerator.
We have created a monster! No, just kidding. We have created a new simple function $g(x)$, which completely captures the sign of $f'(x)$:

$$g(x) = (x - 1)(x - 4)(x + 2)(x + 8).$$

To see where $g(x)$ is positive and where it is negative, simply arrange all zeros $x = 1, 4, -2, -8$ of $g(x)$ on the number line and note that $g(x)$ is a **polynomial** of degree 4 with **positive** leading coefficient (in our case, the leading term is x^4). Then, start from the very right top corner of the graph paper, go down to the first zero of $g(x)$ ($x = 4$ in our case), and then start alternating going above and below the x–axis between the zeros (draw the graph!) We conclude that $f'(x) > 0$ when $x \in (-\infty, -8) \cup (-2, 1) \cup (4, \infty)$, and $f'(x) < 0$ when $x \in (-8, -2) \cup (1, 4)$.

(d) How about the potential local min's and max's? These were at $x = 1, -7, 4$. A **common mistake** is to include here $x = -8, -2$, but ... these are zeros of $g(x)$, and we **don't care** where $g(x) = 0$! $g(x)$ exists with one and only one purpose: to tell us where $f'(x)$ is positive or negative, but **not** where $f'(x) = 0$! So, be very careful here!
OK, at $x = 1$ f' changes sign from $+$ to $-$, hence $f(x)$ has a local maximum at $x = 1$; at $x = 4$ f' changes sign from $-$ to $+$, hence $f(x)$ has a local minimum at $x = 4$. However, at $x = 7$ $f'(x)$ doesn't change sign: it stays negative, so no local extrema at $x = 7$ - the function $f(x)$ simply levels off at $x = 7$ but continues to decrease there.

(e) Let's summarize. $f(x)$ has several interesting points: $x = 1$ (local maximum), $x = 4$ (local minimum), $x = -8, -2, 2$ (vertical asymptotes). Even without having studied yet $f''(x)$, nor having yet found any slant or horizontal asymptotes of $f(x)$, one can draw an approximate graph of $f(x)$ (do it!) To check that $f(x)$ indeed has the concavities showing up in your graph and to find the inflection points, one needs to look at $f''(x)$. Note that one needs to evaluate the actual $f(1)$ and $f(4)$ in the specific problem.

4. Useful Formulas and Miscallaneous Facts

(1) **Manipulations with Fractions**

(a) *Splitting fractions:* $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$; $\frac{ab}{cd} = \frac{a}{c} \cdot \frac{b}{d}$;

(b) **Wrong** *formula:* $\frac{a}{b+c} \neq \frac{a}{b} + \frac{a}{c}$.

(c) *Putting fractions under a common denominator:* $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.

(d) *When denominators have something in common:* $\frac{a}{be} + \frac{c}{de} = \frac{ad+bc}{bde}$.

(e) *"Fractions over fractions":*

$$\frac{a}{c} : \frac{b}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}; \ \frac{a}{\frac{c}{d}} = \frac{ad}{c}; \ \frac{\frac{a}{b}}{c} = \frac{a}{bc}.$$

I cannot conceive of any other operation on fractions! If you think of one, let me know!

(2) **Manipulations with Exponentials and Logarithms**

(a) $a^{b+c} = a^b \cdot a^c$, $\frac{a^b}{a^c} = a^{b-c}$, $(a^b)^c = a^{bc}$, $a^{\frac{b}{c}} = \sqrt[c]{a^b}$, $\frac{1}{\sqrt[c]{a^b}} = \frac{1}{a^{\frac{b}{c}}} = a^{-\frac{b}{c}}$, $a^0 = 1$.

(b) $\ln(e^x) = x$, $e^{\ln x} = x$, $\ln(a^b) = b \ln a$.

(3) **Trigonometric Formulas**

(a) $\sin^2 x + \cos^2 x = 1$;

(b) $\tan x = \frac{\sin x}{\cos x}$; $\cot x = \frac{\cos x}{\sin x}$;

(c) $\sin(\frac{\pi}{2} - x) = \cos x$; $\cos(\frac{\pi}{2} - x) = \sin x$;

(d) $\sin(x+y) = \sin x \cos y + \cos x \sin y$; $\cos(x+y) = \sin x \cos y - \cos x \sin y$.

(e) the values of $\sin x$, $\cos x$, $\tan x$ and $\cot x$ at all "prominent" x's: $0, \pi/6, \pi/4, \pi/3, \pi/2, \pi$, etc.

5. Cheat Sheet

For the midterm, you are allowed to have a "cheat sheet" - *one page* (that is, only on one side of the sheet) of a regular 8×11 sheet. You can write whatever you wish there, under the following conditions:

- The whole cheat sheet must be **handwritten by your own hand**! No xeroxing, no copying, (and for that matter, no tearing pages from the textbook and pasting them onto your cheat sheet.)
- Any violation of these rules will disqualify your cheat sheet and may end in your own disqualification from the midterm. I may decide to randomly check your cheat sheets, so let's play it fair and square. :)
- Don't be a **freakasaurus**! Start studying for the exam several days in advance, and prepare your cheat sheet at least 2 days in advance. This will give you enough time to become familiar with your cheat sheet and be able to use it more efficiently on the exam.