

Topics for Review for Midterm I in Calculus 1A

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1. DEFINITIONS

Be able to **write** precise definitions for any of the following concepts (where appropriate: both in words and in symbols), to **give** examples of each definition, to **prove** that these definitions are satisfied in specific examples. Wherever appropriate, be able to **graph** examples for each definition. What is/are:

- (1) a *function*? an *independent* variable; a *dependent* variable?
- (2) a *domain*; a *range* of a function?
- (3) the *graph* of a function? the vertical line test?
- (4) a *piecewise defined* function?
- (5) the function *absolute value* $|x|$?
- (6) an *increasing* function; a *decreasing* function?
- (7) an *odd* function; an *even* function?
- (8) a *linear*, *quadratic*, *polynomial*, *rational* function? domains of these functions?
- (9) a *power*, *exponential*, *logarithmic* function? domains of these functions?
- (10) a *trigonometric* function: $\sin x$, $\cos x$, $\tan x$, $\cot x$? domains of these functions?

Example. The function $\cot x$ is defined as follows: take a right triangle ABC with $\angle A = x$ (in radians) and $\angle B = \pi/2$; then set $\cot x = AB/BC$ (adjacent leg/opposite leg). Alternatively, let an angle of measure x be centered at the origin O and its first arm coincide with the positive direction of the x -axis. Then $\cot x$ is the horizontal displacement on the line $y = 1$ made by the second arm of the angle. (Check your session notes to review this second definition!)

- (11) the *composition* $f \circ g$ of two functions $f(x)$ and $g(x)$?
- (12) *Limit* of $\lim_{x \rightarrow \square_1} f(x) = \square_2$ where each of the “boxes” can be a finite number or $\pm\infty$?

Example. To say that $\lim_{x \rightarrow -\infty} f(x) = 10$ means that:

- (a) $f(x)$ can be made as close to 10 as we please, provided x is small enough.
 - (b) every ϵ -goal around 10 can be achieved by $f(x)$ provided x is small enough.
 - (c) $\forall \epsilon > 0$, $f(x)$ falls within ϵ of 10 provided x is small enough.
 - (d) $\forall \epsilon > 0 \exists M < 0$ such that $|f(x) - 10| < \epsilon$ whenever $x < M$.
- (13) *One-sided limit*, e.g. $\lim_{x \rightarrow a^-} f(x) = L$, $\lim_{x \rightarrow a^+} f(x) = -\infty$, etc.?

Example. To say that $\lim_{x \rightarrow 4^+} f(x) = 7$ means that

- (a) $f(x)$ can be made as close to 7 as we please, provided x is close enough to 4 and $x > 4$.
 - (b) every ϵ -goal around 7 can be achieved by $f(x)$ provided x is close enough to 4 and $x > 4$.
 - (c) $\forall \epsilon > 0$, $f(x)$ falls within ϵ of 7 provided x is close enough to 4 and $x > 4$.
 - (d) $\forall \epsilon > 0 \exists \delta > 0$ such that $|f(x) - 7| < \epsilon$ whenever $4 < x < 4 + \delta$.
- (14) *Vertical asymptote* of $f(x)$ at a ? *Horizontal asymptote* of $f(x)$?
 - (15) *Continuity*. What does it mean that a function $f(x)$:
 - (a) is *continuous* at a ?
 - (b) is *continuous from the right* (or *from the left*) at a ?
 - (c) is *continuous on* (a, b) , $[a, b)$, $[a, b]$, $(-\infty, b]$, (a, ∞) , etc.
 - (d) has a *removable discontinuity* at a ? *jump discontinuity* at a ? *infinite discontinuity* at a ?
 - (16) the *tangent line* to the graph of a function $f(x)$ at $x = a$? a *secant line* of the graph of $f(x)$? what does it mean that $f(x)$ has a *vertical tangent* at $x = a$?
 - (17) the *average velocity* and the *instantaneous velocity* of an object whose movement is given by $f(t)$?
 - (18) the *derivative* $f'(a)$ at $x = a$? What does it mean that $f(x)$ is *differentiable* at a ?
 - (19) the *derivative function* $f'(x)$? What does it mean that $f(x)$ is *differentiable* on (A, B) ?

2. THEOREMS

Be able to **write** what each of the following theorems (laws, propositions, corollaries, etc.) says. Be sure to understand, distinguish and **state** the conditions (hypothesis) of each theorem and its conclusion. Be prepared to **give** examples for each theorem, and most importantly, to **apply** each theorem appropriately in problems. The latter means: decide which theorem to use, check (in writing!) that all conditions of your theorem are satisfied in the problem in question, and then state (in writing!) the conclusion of the theorem using the specifics of your problem.

- (1) The *Squeeze (Sandwich, Policemen)* Theorem.
- (2) The *Intermediate Value* Theorem (IVT)

Example. The IVT says the following.¹ Let $f(x)$ be a function defined and continuous on an interval $[A, B]$. Let N be an intermediate value for $f(x)$, i.e. N is between $f(A)$ and $f(B)$:

$$f(A) \leq N \leq f(B) \text{ or } f(A) \geq N \geq f(B).$$

Then somewhere on (A, B) the function $f(x)$ attains the value N , i.e. there exists some $x_0 \in (A, B)$ such that $f(x_0) = N$.

- (3) (*Finite*) *Limits Laws (LLs)* (Section 2.3): addition, subtraction, multiplication, division, basic examples, multiplication by a constant, powers, roots. (Be careful about the division law! What extra conditions does it require?) $x \rightarrow \square$ means: $x \rightarrow a$, $x \rightarrow \infty$, or $x \rightarrow -\infty$.

#	Theorem	Hypothesis	Conclusion
1	Linearity	$\lim_{x \rightarrow \square} f_1(x) = L_1, \lim_{x \rightarrow \square} f_2(x) = L_2, c_1, c_2 \in \mathbb{R}$	$\lim_{x \rightarrow \square} (c_1 f_1(x) + c_2 f_2(x)) = c_1 L_1 + c_2 L_2$
2	Product	$\lim_{x \rightarrow \square} f_1(x) = L_1, \lim_{x \rightarrow \square} f_2(x) = L_2$	$\lim_{x \rightarrow \square} f_1(x) f_2(x) = L_1 L_2$
3	Quotient	$\lim_{x \rightarrow \square} f_1(x) = L_1, \lim_{x \rightarrow \square} f_2(x) = L_2, f_2(x) \neq 0$ for $x \approx a, L_2 \neq 0$	$\lim_{x \rightarrow \square} \frac{f_1(x)}{f_2(x)} = \frac{L_1}{L_2}$
4	∞ -Limit Sum	$\lim_{x \rightarrow \square} f_1(x) = \infty$, and $\lim_{x \rightarrow \square} f_2(x) = \infty$ or $f_2(x)$ bounded below	$\lim_{x \rightarrow \square} (f_1(x) + f_2(x)) = \infty$
5	∞ -Limit Product	$\lim_{x \rightarrow \square} f_1(x) = \infty$, and $\lim_{x \rightarrow \square} f_2(x) = L > 0$ or $f_2(x)$ bounded below by $K > 0$	$\lim_{x \rightarrow \square} f_1(x) f_2(x) = \infty$
6	∞ -Reciprocal	$\lim_{x \rightarrow \square} f(x) = \infty$	$\lim_{x \rightarrow \square} \frac{1}{f(x)} = 0$
7	0-Reciprocal	$\lim_{x \rightarrow \square} \frac{1}{f(x)} = 0$ and $f(x) > 0$	$\lim_{x \rightarrow \square} f(x) = \infty$
8	Sandwich, Policemen	$f(x) \leq g(x) \leq h(x)$ for $x \approx \square$ and $\lim_{x \rightarrow \square} f(x) = L = \lim_{x \rightarrow \square} h(x)$	$\lim_{x \rightarrow \square} g(x) = L$
9	∞ -Sandwich	$f(x) \leq g(x)$ for $x \approx \square$ and $\lim_{x \rightarrow \square} f(x) = \infty$	$\lim_{x \rightarrow \square} g(x) = \infty$
10	Lim-Location	$f(x) \leq M$ for $x \approx \square$ and $\lim_{x \rightarrow \square} f(x)$ exists,	$\lim_{x \rightarrow \square} f(x) \leq M$
11	Lim-Location	$f(x) \leq g(x)$ for $x \approx \square, \exists \lim_{x \rightarrow \square} f(x), \exists \lim_{x \rightarrow \square} g(x)$	$\lim_{x \rightarrow \square} f(x) \leq \lim_{x \rightarrow \square} g(x)$
12	Fun-Location	$\lim_{x \rightarrow \square} f(x) = L < M$	$f(x) < M$ for $x \approx \square$

¹In order to apply IVT, one must first check (and write) the two conditions: that $f(x)$ is continuous on $[A, B]$, and that N is indeed an intermediate value between $f(A)$ and $f(B)$; only after that, one states the conclusion of the IVT for the particular $f(x)$ in question.

(4) *Infinite Limit Laws (∞ -LLs).*

Name	∞ -LL: Formula	Example
1. addition	$\infty + \infty = \infty$	$\lim_{x \rightarrow \infty} (x + x^2) = \lim_{x \rightarrow \infty} x + \lim_{x \rightarrow \infty} x^2 = \infty + \infty \stackrel{\infty\text{LL}}{=} \infty$
	$(-\infty) + (-\infty) = -\infty$	$\lim_{x \rightarrow \infty} (-x - x^2) = \lim_{x \rightarrow \infty} (-x) + \lim_{x \rightarrow \infty} (-x^2) = (-\infty) + (-\infty) \stackrel{\infty\text{LL}}{=} -\infty$
	$\infty - \infty$ undefined	Never use ∞-LLs in such cases.
2. multipli- cation	$\infty \cdot \infty = \infty$	$\lim_{x \rightarrow \infty} x^2 = \lim_{x \rightarrow \infty} x \cdot \lim_{x \rightarrow \infty} x = \infty \cdot \infty \stackrel{\infty\text{LL}}{=} \infty$
	$(-\infty) \cdot (-\infty) = \infty$	$\lim_{x \rightarrow \infty} (-x)^2 = \lim_{x \rightarrow \infty} (-x) \cdot \lim_{x \rightarrow \infty} (-x) = (-\infty) \cdot (-\infty) \stackrel{\infty\text{LL}}{=} \infty$
	$\infty \cdot (-\infty) = -\infty$	$\lim_{x \rightarrow \infty} -x^2 = \lim_{x \rightarrow \infty} x \cdot \lim_{x \rightarrow \infty} (-x) = \infty \cdot (-\infty) \stackrel{\infty\text{LL}}{=} -\infty$
3. multipli- cation by constant	$c \cdot \infty = +\infty$ if $c > 0$	$\lim_{x \rightarrow \infty} (2x) = 2 \lim_{x \rightarrow \infty} x = 2 \cdot \infty \stackrel{\infty\text{LL}}{=} \infty$
	$c \cdot \infty = -\infty$ if $c < 0$	$\lim_{x \rightarrow \infty} (-2x) = -2 \lim_{x \rightarrow \infty} x = -2 \cdot \infty \stackrel{\infty\text{LL}}{=} -\infty$
	$0 \cdot \infty$ undefined	Never use ∞-LLs in such cases.
4. addition by constant	$c + \infty = +\infty \forall c$	$\lim_{x \rightarrow \infty} (-2 + x) = -2 + \lim_{x \rightarrow \infty} x = -2 + \infty \stackrel{\infty\text{LL}}{=} \infty$
	$c - \infty = -\infty \forall c$	$\lim_{x \rightarrow \infty} (-2 - x) = -2 - \lim_{x \rightarrow \infty} x = -2 - \infty \stackrel{\infty\text{LL}}{=} -\infty$
5. division by $\pm\infty$	$\frac{c}{+\infty} = 0 \forall c$	$\lim_{x \rightarrow \infty} \frac{2}{x} = \frac{2}{\lim_{x \rightarrow \infty} x} = \frac{2}{\infty} \stackrel{\infty\text{LL}}{=} 0$
	$\frac{c}{-\infty} = 0 \forall c$	$\lim_{x \rightarrow \infty} \frac{2}{-x} = \frac{2}{\lim_{x \rightarrow \infty} (-x)} = \frac{2}{\infty} \stackrel{\infty\text{LL}}{=} 0$
6. division by 0^\pm	$\frac{c}{0^+} = +\infty \forall c > 0$	$\lim_{x \rightarrow 0} \frac{2}{x^2} = \frac{2}{0^+} \stackrel{\infty\text{LL}}{=} +\infty, \lim_{x \rightarrow 0} \frac{-2}{x^2} = \frac{-2}{0^+} \stackrel{\infty\text{LL}}{=} -\infty$
	$\frac{c}{0^-} = -\infty \forall c > 0$	$\lim_{x \rightarrow 0} \frac{2}{-x^2} = \frac{2}{0^-} \stackrel{\infty\text{LL}}{=} -\infty, \lim_{x \rightarrow 0} \frac{-2}{-x^2} = \frac{-2}{0^-} \stackrel{\infty\text{LL}}{=} +\infty$
	$\frac{0}{0}, \frac{c}{0}$ undefined $\forall c$	Never use ∞-LLs in such cases.
7. basic cases	$\lim_{x \rightarrow \infty} \frac{1}{x} = 0, \lim_{x \rightarrow -\infty} \frac{1}{x} = 0$	
	$\lim_{x \rightarrow \infty} x = \infty, \lim_{x \rightarrow -\infty} x = -\infty$	
	$\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty, \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty, \lim_{x \rightarrow 0} \frac{1}{x}$ does not exist.	

In the infinite limit laws, an expression like “ $(-\infty) + (-\infty) = -\infty$ ” does **not** have a meaning on its own, except *in context*, i.e. it refers only to the following situation and to nothing else:

Theorem “ $(-\infty) + (-\infty) = -\infty$ ”. “If for functions $f(x)$ and $g(x)$ we know that $\lim_{x \rightarrow \square} f(x) = -\infty$, $\lim_{x \rightarrow \square} g(x) = -\infty$, then $f(x) + g(x)$ also has a limit when $x \rightarrow \square$: this limit is $\lim_{x \rightarrow \square} (f(x) + g(x)) = -\infty$.”

Note that there are no infinite limit laws for subtraction, division, or multiplication of the type $0 \cdot \infty$, i.e. the symbolic expressions $\infty - \infty, \infty/\infty, 0 \cdot \infty$ do not make sense, and they are called *indeterminate*.

(5) *Continuity Laws (CLs)*. Hypothesis for all continuity theorems below: If $f(x)$ and $g(x)$ are *continuous* at $x = a$, i.e. $\lim_{x \rightarrow a} f(x) = f(a)$ and $\lim_{x \rightarrow a} g(x) = g(a)$, then

#	Theorem Name	Conclusion	Follows from
1	Continuity of Sum	$f(x) + g(x)$ is also continuous at $x = a$	LL for sum
2	Continuity of Difference	$f(x) - g(x)$ is also continuous at $x = a$	LL for difference
3	Continuity of Product	$f(x)g(x)$ is also continuous at $x = a$	LL for product
4	Continuity of Ratio ($g(a) \neq 0$)	$f(x)/g(x)$ is also continuous at $x = a$	LL for ratio
5	Continuity Respects Constants	$c \cdot f(x)$ is also continuous at $x = a$	LL for jumping constants
6	Continuity of Composition: $h(x)$ is continuous at $b = f(a)$	$h(f(x))$ is also continuous at $x = a$	LL for composition
7	Continuity & Positivity: $f(a) > 0$	$f(x) > 0$ for $x \approx a$	Limit Definition for $f(x)$

Note that all *Continuity Laws (CLs)* follow from the corresponding *Limit Laws (LLs)*. The CLs above allow us to perform algebraic operations (and compositions) on continuous functions. Thus,

we can construct more complex continuous functions from simpler continuous functions. To do this, we need to have a starting collection of

- (6) *Basic Continuous Functions.* All of the following types of functions are *continuous* on their respective domains of definition:

#	Function	Algebraic Formula and Conclusion	Follows from
1	Constants	c continuous at $\forall x$	LL for constants
2	Linear	$ax + b$ continuous at $\forall x$	LL for linear fn's
3	Quadratic	$ax^2 + bx + c$ continuous at $\forall x$	LL for quadratic fn's
4	Power	x^n continuous at $\forall x, \forall n = 1, 2, 3, \dots$	LL for powers
5	Polynomial	$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ continuous at $\forall x$	LL for polynomials
6	Rational	$\frac{f(x)}{g(x)}$ continuous where $g(x) \neq 0$ ($f(x), g(x)$ - poly's)	LL for ratio, CL for poly's
7	Root	$\sqrt[n]{x}$ continuous at $\forall x$ where defined	LL for roots
8	Exponential	a^x continuous at $\forall x$ ($a > 0$)	LL for exponentials
9	Logarithmic	$\log_a x, \ln x$ continuous at $\forall x > 0$ ($a > 0$)	LL for logarithmics
10	Trigonometric	$\sin x, \cos x$ continuous at $\forall x$; $\tan x, \cot x$ cont. on domain: $\tan x: x \neq \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \dots, (2n+1)\pi/2,$ $\cot x: x \neq 0, \pm\pi, \pm 2\pi, \pm 3\pi, \dots, n\pi, n \in \mathbb{Z}$	LL for trig. fn's
11	Inverse Trigonometric	$\arcsin x, \arccos x, \operatorname{arccot} x, \operatorname{arccot} x$ continuous on domain: $\arcsin x: [-1, 1] \rightarrow [-\pi/2, \pi/2], \arccos x: [-1, 1] \rightarrow [0, \pi],$ $\operatorname{arccot} x: \mathbb{R} \rightarrow (-\pi/2, \pi/2), \operatorname{arccot} x: \mathbb{R} \rightarrow (0, \pi)$	LL for inverse trig. fn's

- (7) *Special Continuity Theorems:* continuity statements about various types of functions, e.g. polynomials, rational, root, trigonometric, exponential and logarithmic functions. (Be careful to include the domain of definition of these functions.)

Example. The *logarithmic* function $\log_c x$ is defined for any $x > 0$ (the base of the logarithm, c , must also be positive.) Any logarithmic function is continuous everywhere on its domain of definition, i.e. on $(0, +\infty)$.

- (8) *“Double-sided” Theorems for Limits and Continuity.*

Example. The limit of $f(x)$ when $x \rightarrow a$ exists if and only if the two one-sided limits of $f(x)$ at a exist and are equal to each other. In this case, the limit of $f(x)$ exists and equals their common value. In symbols: $\lim_{x \rightarrow a} f(x) = L$ if and only if $\lim_{x \rightarrow a^+} f(x) = L = \lim_{x \rightarrow a^-} f(x)$.

- (9) *Theorem I. (Differentiable \Rightarrow continuous.)* If $f(x)$ is differentiable at a , then $f(x)$ is continuous at a . If $f(x)$ is differentiable everywhere on its domain, then it is also continuous everywhere on its domain.

- (10) *Contrapositive Theorem II. (Non-differentiable \Rightarrow non-continuous.)* If $f(x)$ is *not* continuous at a , then $f(x)$ is *not* differentiable at a .

- (11) *Converse Statement is False!* Continuity does not guarantee differentiability. Counterexample?

- (12) *Differentiation Laws (DLs) allowed on Midterm 1*

(a) *Constant Functions:* $(c)' = 0$ for any constant c .

(b) *Power Rule:* $(x^c)' = cx^{c-1}$ for any constant c .

(c) *Multiplication by a Constant:* If $f(x)$ is a differentiable function, then $(cf(x))' = cf'(x)$.

(d) *Sum and Difference Rules:* If $f(x)$ and $g(x)$ are differentiable, then their sum and difference are also differentiable: $(f(x) + g(x))' = f'(x) + g'(x)$, and $(f(x) - g(x))' = f'(x) - g'(x)$.

3. PROBLEM SOLVING TECHNIQUES

- (1) **How do we find $\lim_{x \rightarrow a} f(x)$ when the (finite) LLs fail?**
- (a) If $f(x) = \frac{g(x)}{h(x)}$ and “plugging in a ” yields $\frac{0}{0}$, try factoring polynomials and rationalizing expressions with square roots. The idea is to end up with $(x - a)$ both in numerator and denominator, cancel it, and then again attempt to apply LLs.
 - (b) If $f(x)$ is a piecewise-defined function (i.e. given by different formulas on several intervals), try first to find the left-hand and the right-hand limits separately, and then compare them to see if they are equal (or if they exist, for that matter).
 - (c) If $f(x)$ is given by a formula involving absolute values, again proceed by finding and comparing the two one-sided limits.
- (2) **How do we determine if a function is continuous at a ?** By definition of continuity, there are 3 things to check:
- (a) Find $\lim_{x \rightarrow a} f(x)$ by following either limits laws or the techniques suggested above. (If it doesn't exist, then the function has no chance of being continuous at a . If it exists but is an infinite limit $\pm\infty$, again the function is not continuous at a ; in fact, it has an *infinite discontinuity* at a .)
 - (b) Find $f(a)$. If f is not defined at a , then the function is not continuous at a .
 - (c) If the above two steps yield two finite numbers, compare them to check if they are equal: $\lim_{x \rightarrow a} f(x) = f(a)$. If yes, the function is continuous at a ; if not, the function is not continuous at a .
- (3) **How do we determine if a function is continuous at a without using the definition of continuity?** We use CLs if applicable. For example, the function $f(x) = \frac{1}{x-3} \cdot \cos x + 6x^3$ is continuous at 2 because all comprising functions (rational, trigonometric and polynomial) are all continuous at 2. However, the function is not continuous at 3. (Why?)
- (4) **How do we find $\lim_{x \rightarrow \square_1} f(x)$ when infinite limits are involved, but ∞ -LLs fail?**
- (a) When finding the limit of a rational function: $\lim_{x \rightarrow \pm\infty} \frac{P_1(x)}{P_2(x)}$ (here $P_1(x)$ and $P_2(x)$ are polynomials), we know that ∞/∞ doesn't make sense. So, we factor out the highest powers of x from both top and bottom polynomials, cancel, and then apply LLs again. (Note: In the end, all that will matter will be the leading terms of the two polynomials - no other terms will survive the above operations.) Similar ideas apply to any other fractions which involve polynomials and possibly radicals.
 - (b) If ∞ -LLs produce expressions involving $\infty - \infty$, we know that this doesn't make sense, so we look for a different approach. If polynomials are involved, factoring out the highest power is a good start. If square roots are involved, rationalizing might help. If two or more fractions are involved, putting them under a common denominator to arrive at one single fraction is the first step; then apply other techniques mentioned above.
 - (c) If ∞ -LLs produce an expression of the type $0 \cdot \infty$, we know that this doesn't make sense, so we look for a different approach. Each example of this type has to be considered individually; most likely, we will end up factoring or rationalizing in search of common things to cancel, and after that we will attempt again to apply LLs.
- (5) **How do we sketch graphs of functions $f(x)$?**

- (a) If you recognize that the graph of $f(x)$ can be obtained from a graph of a well-known function via horizontal and/or vertical shifts, go for it! If the original (well-known) function already has vertical or horizontal asymptotes, make sure to shift them too and indicate that the resulting function has these-and-these asymptotes.
- (b) If the graph of $f(x)$ can't be obtained via the above shifts (e.g. $f(x)$ is a complicated function, or you just forgot how to sketch the graph of the "well-known" function), proceed as follows:
- (i) First, look for "zeros" of the functions, i.e. for its x -intercepts: try to solve $f(x) = 0$ if possible. If $f(x)$ is a fraction, such solutions will be produced in the numerator. (The denominator will be irrelevant in this step.) It is also good to find the y -intercept, by setting $x = 0$ in $f(x)$.
 - (ii) Second, look for vertical asymptotes: these will appear where $f(x)$ has an infinite (at least) one-sided limit, i.e. if $\lim_{x \rightarrow a^+} f(x) = \pm\infty$ or $\lim_{x \rightarrow a^-} f(x) = \pm\infty$; then the vertical line $x = a$ is such an asymptote. If $f(x)$ is a fraction, such solutions will be produced by the roots of the denominator. (The numerator will be irrelevant in this step.)
 - (iii) Third, look for horizontal asymptotes: these will appear where $f(x)$ has a finite limit when $x \rightarrow \pm\infty$, i.e. if $\lim_{x \rightarrow \pm\infty} f(x) = L$; then the horizontal line $y = L$ is such an asymptote. If $f(x)$ is a fraction, both numerator and denominator will be involved in this step.
 - (iv) Finally, draw the vertical and horizontal asymptotes, mark the x -intercepts (and the y -intercept if applicable); draw the function so that it passes through the x - and y -intercepts and respects all asymptotes as found above. Be careful nearby the vertical asymptotes to reflect whether a given one-sided limit is $+\infty$ or $-\infty$, correspondingly. It doesn't hurt to plot several other points in each interval between asymptotes and x -intercepts, to make your graph more precise and make sure you haven't done any silly calculation mistakes in the above steps.

(6) How do we prove statements about limits using the limit definitions?

We pray that such a problems is not on the midterm. If this doesn't help, we follow the steps below.

- (a) Consider the type of limit you are given: $\lim_{x \rightarrow \square_1} f(x) = \square_2$ and decide what your goals will be depending on what \square_2 is.
- If $\square_2 = L$ - a finite number, then your function "wants to be close to this number L " ... how close? - ϵ -close. Your goals will be therefore ϵ -goals around L , which can be written in one of the following three equivalent forms:

$$f(x) \text{ is within } \epsilon \text{ of } L \Leftrightarrow L - \epsilon < f(x) < L + \epsilon \Leftrightarrow |f(x) - L| < \epsilon$$
 - If $\square_2 = +\infty$, then your function "wants to be close to $+\infty$ "... To say " ϵ -close to $+\infty$ " doesn't make a whole lot of sense! Instead, we want $f(x)$ to get as large as we please. Hence, our goals will be M -goals, where $M > 0$. Such a goal can be written simply as $f(x) > M$.
 - If $\square_2 = -\infty$, then your function "wants to be close to $-\infty$ "... To say " ϵ -close to $-\infty$ " doesn't make a whole lot of sense! Instead, we want $f(x)$ to get as small as we please. Hence, our goals will be M -goals, where $M < 0$. Such a goal can be written simply as $f(x) < M$.

- (b) Decide next what type of answers you are looking for depending on what \square_1 is.

- If $\square_1 = a$ - a finite number, then x "wants to be close to this number a " ... how close? - δ -close. Your answers will be therefore δ -intervals around a , which can be written in one of the following three equivalent forms:

$$x \text{ is within } \delta \text{ of } a \Leftrightarrow a - \delta < x < a + \delta \Leftrightarrow |x - a| < \delta.$$

- If $\square_1 = +\infty$, then x “wants to be close to $+\infty$ ”... To say “ δ -close to $+\infty$ ” doesn’t make a whole lot of sense! Instead, we want x to be large enough. Hence, our answers will be M -answers, where $M > 0$, written simply as $x > M$.
- If $\square_1 = -\infty$, then x “wants to be close to $-\infty$ ”... To say “ δ -close to $-\infty$ ” doesn’t make a whole lot of sense! Instead, we want x to be small enough. Hence, our answers will be M -answers, where $M < 0$, written simply as $x < M$.
- **Summary of (ϵ, δ) –Definition Types of Goals and Answers.** Here are all 9 possible types of limits (3 possible goals, and 3 possible answers) for $\lim_{x \rightarrow \square_1} f(x) = \square_2$:

limit \square_2	goal for $f(x)$	$x \rightarrow \square_1$	answer for x
L	ϵ -goal around L	$x \rightarrow a$	δ -interval around a
$+\infty$	M -goal, $M > 0$	$x \rightarrow +\infty$	N -answer, $N > 0$
$-\infty$	M -goal, $M < 0$	$x \rightarrow -\infty$	N -answer, $N < 0$

(c) Put together your goals and your types of answers to see what you are really after. And then proceed either algebraically or graphically.

- (i) The algebraic way is more rigorous, but then less insightful. It goes as follows.

Step I: Conjecture your limit.

Step II: Write your goal for your function, e.g. we want $5 - \epsilon < f(x) < 5 + \epsilon$ (here $\square_2 = L = 5$), or $f(x) < M$ (here $\square_2 = -\infty$ and $M < 0$). Next, try to solve these inequalities for x . Keep in mind that sometimes considerations like $x > 0$ or $x < 0$ can help eliminate irrelevant information. Next, put your answer in the form of δ -interval around a , or in the form of $x > M$ or $x < M$, depending on what type of answer you are looking for. (Recall that sometimes when looking for δ , we have to take the smaller of two distances from a in order to ensure that our δ -interval is centered at a .) Make sure you announce what your final δ or M answer is, do NOT forget to state your conclusion: The above steps show that indeed blah-blah ... ($\lim_{x \rightarrow \square_1} f(x) = \square_2$)

- (ii) The geometric way is less rigorous, but then more intuitive. Start by sketching a graph of your function (you may have to shift graphs of simpler functions or plot several points for your function). Then mark your goal: either an ϵ -strip around L , or an M -region (above or below M depending on whether your limit is $+\infty$ or $-\infty$.)

Mark the portion of the graph which lies inside your goal area. Project this portion onto the x -axis, i.e. mark all x 's above which $f(x)$ falls into the goal area. These x 's should be grouped in one or more intervals. Make sure you eliminate the irrelevant intervals for x (e.g. if $x \rightarrow -\infty$, then intervals with positive x 's are irrelevant; if $x \rightarrow 7$, then an interval around 2 is probably also irrelevant, but an interval around 7 will be VERY relevant!) You should be left with only one relevant interval I for x .

Using the formula for the function $f(x)$, find precisely what this I is. This usually entails the following calculations: $f(x) = L + \epsilon$ (when the function enters/exits “from above” the ϵ -strip around L); or $f(x) = L - \epsilon$ (when the function enters/exits “from below” the ϵ -strip around L); or $f(x) = M$ (when the function enters/exits the M -goal region). Solve this for x to find your “good interval” I - check with your graph to make sure that what you obtain makes sense.

Finally, translate this into the type of answer you are expected to obtain: $a - \delta < x < a + \delta$, or $x < M$ or $x > M$, and state this answer clearly. (Make sure that all of the above calculations and work with the graph is recorded properly in your solution!) Conclude by stating that therefore blah-blah... ($\lim_{x \rightarrow \square_1} f(x) = \square_2$)

(7) **How do we find derivatives from the definition?** Read carefully if you are being asked to find a specific derivative $f'(a)$, or the *whole* derivative function $f'(x)$. In each case, you have two choices how to proceed, as listed below.

(a) $f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$. Here a is a constant and x moves towards a , so we expect that x will disappear and a will remain in the final result for $f'(a)$.

(b) $f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$. Here a is a constant and h moves towards 0, so we expect that h will disappear and a will remain in the final result for $f'(a)$. This formula is nothing else but formula (a) where x is replaced by $a+h$.

(c) $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$. Here x is viewed as a constant and h moves towards 0, so we expect that h will disappear and x will remain in the final result for the derivative function $f'(x)$. This formula is nothing else but formula (b) where a is replaced by x .

(d) One can also find $f'(x)$ by first finding $f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$; in the result of this calculation x will disappear and only a will remain; in this final formula replace a by x to obtain a formula for the whole derivative function $f'(x)$.

(8) **How do we find equations of tangent lines?**

(a) First find the corresponding derivative $f'(a)$: this will be the slope of your tangent line.

(b) Next use the point-slope formula for the point $P(a, f(a))$ and for the slope $f'(a)$ from part (a):

$$(1) \quad f'(a) = \frac{y - f(a)}{x - a} \Leftrightarrow y - f(a) = f'(a)(x - a).$$

Here a and $f(a)$ are constants, and x and y are variables in the equation for your tangent line. Where suitable, multiply through and simplify to obtain a formula of the type:

$$y = f'(a) \cdot x + b.$$

(9) **How do we find all the tangents lines to the graph of $f(x)$ which are parallel to some line $y = mx + b$?**

(a) First find the slope m of the given line; be careful with this since the line may not be given in the standard “line-equation” form as above and you may have to rewrite it.

(b) Find the derivative $f'(x)$.

(c) Next, set $f'(x) = m$, where m is the found slope of the line above, and solve it for x .

(d) Finally, let's say you found several solutions x_1, x_2, \dots etc. What remains is to find the corresponding points on the graph of $f(x)$ through which the wanted tangent lines will pass: $P_1(x_1, f(x_1)), P_2(x_2, f(x_2)), \dots$ etc. If the problem asks for finding the equation of these tangent lines, well, proceed now with the point-slope formula as before.

(10) **How do we find all tangent lines to the graph of $f(x)$ passing through a point (c, d) ?**

(a) First find the derivative $f'(x)$.

(b) Next, set the point-slope formula for the tangent line:

$$f'(a) = \frac{y - f(a)}{x - a}.$$

You will not know at this moment what a is (that's what you want to find in the end), nor what x or y are (these are the variables in the equation of the tangent line).

- (c) Substitute the point (c, d) into the above equation ((c, d) is supposed to lie on your tangent line, hence substituting $x = c$ and $y = d$ must work):

$$f'(a) = \frac{d - f(a)}{c - a}.$$

At this point, you must realize that only a is left “unknown” in this equation; everything else must be a number; thus, we can solve this equation for a .

- (d) Now that you know what a is (are), find the equation of the corresponding tangent line(s) at $(a, f(a))$. The problem may be asking for less: just find the points of tangency $(a, f(a))$.

(11) **How do we sketch graphs of the derivative function $f'(x)$ given the graph of $f(x)$?**

- (a) Find where the given function $f(x)$ is *not* differentiable; at these x 's $f'(x)$ will not exist. There are many different reasons for $f'(x)$ not to exist. Here follow some such reasons:

- $f(x)$ is *not* defined at $x = a$. Then we can't even talk about the derivative at $x = a$.
- $f(x)$ is defined at $x = a$ but is *not* continuous there. Then the contrapositive theorem implies that $f(x)$ is *not* differentiable at $x = a$. No matter what type of discontinuity $f(x)$ has at $x = a$, $f'(a)$ will *not* exist. An infinite discontinuity of $f(x)$ (i.e. $f(x)$ has a vertical asymptote $x = a$) usually translates into a vertical asymptote for $f'(x)$ at $x = a$. A jump or removable discontinuity of $f(x)$ usually translates into a jump or removable discontinuity for $f'(x)$. Each case is treated separately to see what happens with $f'(x)$.
- $f(x)$ is defined and continuous at $x = a$, but is *not* “smooth” there, i.e. has a cusp(corner). Usually here either the two one-sided tangents exist at $x = a$ but have different slopes, or there is a vertical tangent at $x = a$. In the former case, this will translate into a jump discontinuity of $f'(x)$; in the latter case, this translates into a vertical asymptote of $f'(x)$.
- $f(x)$ looks “smooth” at $x = a$, but has a vertical tangent there. Again, this will translate into a vertical asymptote of $f'(x)$.

- (b) After marking all x 's where $f'(x)$ does not exist (including possible vertical asymptotes, etc.), we move on to graphing $f'(x)$ where it exists. First find all places where the tangents to $f(x)$ are horizontal and mark the corresponding 0's on the graph of f' . Next determine the intervals where $f(x)$ increases i.e. has positive tangent slopes, and where $f(x)$ decreases, i.e. has negative tangent slopes. In the former case, $f'(x)$ will be positive, and in latter case, $f'(x)$ will be negative. In each such interval, answer the following two questions: whether the tangent slopes are positive or negative, and whether the tangent slopes themselves are increasing or decreasing. Translate this into the corresponding property of $f'(x)$.

- (c) For more precise drawing, in each of the above intervals mark several tangent lines, guesstimate their slopes and mark the corresponding points on the graph of $f'(x)$. Connect all these marked points to obtain the graph of $f'(x)$. Don't forget the places where $f'(x)$ was not defined!

(12) **How do we find derivatives using DLs?** If you are given $f(x)$ via one formula and you are not asked to use the definition of derivative, you apply DLs. However, for the purposes of this Midterm 1, the only DLs allowed are listed previously in this handout: DLs for polynomials, power rule, DLs for addition, difference and multiplication by a constant.

- (a) First see if you can further simplify the given formula. In particular, try to avoid applying the Quotient Rule whenever possible because it is prone to errors. In practice this mean: try to get rid of denominators by either splitting fractions and then simplifying each fraction separately (see formulas for fraction manipulations below), or by direct cancellation of common stuff in the numerator and denominator, or by moving the denominator into the numerator: e.g. x^3 in the denominator becomes x^{-3} in the numerator.

- (b) If you are going to apply the Power Rule, turn all expressions like $\sqrt[n]{x^m}$ into the standard form $x^{\frac{m}{n}}$. Again, such expressions in the denominator should move into the numerator wherever suitable by flipping the sign of the power: $\sqrt[n]{x^m}$ in the denominator becomes $x^{-\frac{m}{n}}$ in the numerator.
- (c) Look at your function $f(x)$ to figure out its components, the simpler pieces it is made of, and decide which DL(s) you are going to use. In some cases, you may have to apply several DLs one after the other, so keep good track of your intermediate results, or else your calculations will be untraceable. A good strategy is to name some of the simpler components of $f(x)$, e.g. $g(x)$, $h(x)$, etc. and perform some of the necessary differentiation on these functions on the side and then put back your results together. To reduce errors and to make clear that you do know the DLs, it is always good to write the DL formula in terms of functions at first, e.g.

$$((5x + 2) \cdot x^3)' \stackrel{PR}{=} (5x + 2)' \cdot x^3 + (5x + 2) \cdot (x^3)' = \dots$$

However, remember that neither Quotient nor Product rules will be allowed on Midterm 1, so you must manipulate your function to a form where more basic (and allowed) DLs are applicable.

4. USEFUL FORMULAS AND MISCELLANEOUS FACTS

- (1) **Quadratic formula:** useful for factoring quadratic polynomials as $a(x - x_1)(x - x_2)$, where x_1 and x_2 are the two roots of the polynomial, and a is the leading coefficient. Useful also for graphing quadratic polynomials: will yield the x -intercepts (or tell you that they don't exist.)

(2) **Rationalizing formula:** $\sqrt{A} - \sqrt{B} = \frac{(\sqrt{A} - \sqrt{B})(\sqrt{A} + \sqrt{B})}{\sqrt{A} + \sqrt{B}} = \frac{(A - B)}{\sqrt{A} + \sqrt{B}}$.

(3) **Factorization formulas:** $A^2 - B^2 = (A - B)(A + B)$ and $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$.

(4) **Binomial formulas:** $(A + B)^2 = A^2 + 2AB + B^2$, $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$.

- (5) **Putting fractions under a common denominator.** The most general formula is as follows: $\frac{A}{B} + \frac{C}{D} = \frac{AD + BC}{BD}$. Yet, it is worth noting that if fractions already share something in their denominators, it will be faster to take this into account, e.g.

$$\frac{2x + 1}{x^2} + \frac{x^3}{x(x - 1)} = \frac{(2x + 1)(x - 1) + x \cdot x^3}{x^2(x - 1)} = \frac{x^4 + 2x^2 - x - 1}{x^2(x - 1)}.$$

- (6) **Absolute value inequalities.** Note the following expressions which mean the same things:

(a) $|x| < A \Leftrightarrow -A < x < A$;

(b) $|x - B| < A \Leftrightarrow -A < x - B < A \Leftrightarrow B - A < x < B + A \Leftrightarrow x \in (B - A, B + A)$;

(c) $|3x^2 - 10| < 5 \Leftrightarrow -5 < 3x^2 - 10 < 5 \Leftrightarrow 5 < 3x^2 < 15 \Leftrightarrow \sqrt{5/3} < |x| < \sqrt{15/3}$

$\Leftrightarrow \sqrt{5/3} < x < \sqrt{15/3}$ (when $x \geq 0$), or $\sqrt{5/3} < -x < \sqrt{15/3}$ (when $x < 0$).

As a final answer, the original inequality is satisfied when $x \in (\sqrt{5/3}, \sqrt{5}) \cup (-\sqrt{5}, -\sqrt{5/3})$.

(d) $|2f(x) - 7| < 0.5 \Leftrightarrow -0.5 < 2f(x) - 7 < 0.5 \Leftrightarrow 3.25 < f(x) < 3.75 \Leftrightarrow f(x) \in (3.25, 3.75)$.

- (7) **Trigonometric functions:** domains of definition, ranges, graphs, periods; where they increase, decrease, values at $x = 0, \pi/3, \pi/4$ and so on "prominent" numbers; vertical asymptotes (if any); trigonometric identities; radians versus degrees.
- (8) **Exponential and Logarithmic functions:** domains of definition, ranges, graphs; for which bases do these function increase/decrease, c^x and $\log_c x$ are inverse functions of each other, basic identities.
- (9) **Manipulations with Fractions**

- (a) *Splitting fractions*: $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$; $\frac{ab}{cd} = \frac{a}{c} \cdot \frac{b}{d}$;
- (b) **Wrong formula**: $\frac{a}{b+c} \neq \frac{a}{b} + \frac{a}{c}$.
- (c) *Putting fractions under a common denominator*: $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.
- (d) *When denominators have something in common*: $\frac{a}{be} + \frac{c}{de} = \frac{ad+bc}{bde}$.
- (e) *“Fractions over fractions”*: $\frac{a}{c} : \frac{b}{d} = \frac{\frac{a}{c}}{\frac{b}{d}} = \frac{ad}{bc}$; $\frac{a}{\frac{c}{d}} = \frac{ad}{c}$; $\frac{\frac{a}{b}}{c} = \frac{a}{bc}$.

I cannot conceive of any other operation on fractions! If you think of one, let me know!

(10) Manipulations with Exponentials and Logarithms

- (a) $a^{b+c} = a^b \cdot a^c$, $\frac{a^b}{a^c} = a^{b-c}$, $(a^b)^c = a^{bc}$, $a^{\frac{b}{c}} = \sqrt[c]{a^b}$, $\frac{1}{\sqrt[c]{a^b}} = \frac{1}{a^{\frac{b}{c}}} = a^{-\frac{b}{c}}$, $a^0 = 1$.
- (b) $\ln(e^x) = x$, $e^{\ln x} = x$, $\ln(a^b) = b \ln a$.

(11) Trigonometric Formulas

- (a) $\sin^2 x + \cos^2 x = 1$;
- (b) $\tan x = \frac{\sin x}{\cos x}$; $\cot x = \frac{\cos x}{\sin x}$;
- (c) $\sin(\frac{\pi}{2} - x) = \cos x$; $\cos(\frac{\pi}{2} - x) = \sin x$;
- (d) $\sin(x+y) = \sin x \cos y + \cos x \sin y$; $\cos(x+y) = \cos x \cos y - \sin x \sin y$.
- (e) the values of $\sin x$, $\cos x$, $\tan x$ and $\cot x$ at all “prominent” x ’s: $0, \pi/6, \pi/4, \pi/3, \pi/2, \pi$, etc.

5. CHEAT SHEET

For the midterm, you are allowed to have a “cheat sheet” - *one page* (which means only on one side) of a regular 8×11 sheet. You can write whatever you wish there, under the following conditions:

- The whole cheat sheet must be **handwritten by your own hand!** No xeroxing, no copying, (and for that matter, no tearing pages from the textbook and pasting them onto your cheat sheet.)
- Any violation of these rules will disqualify your cheat sheet and may end in disqualifying your midterm. I may decide to randomly check your cheat sheets, so let’s play it fair and square. :)
- Don’t be a **freakasaurus!** Start studying for the exam several days in advance, and prepare your cheat sheet at least 2 days in advance. This will give you enough time to become familiar with your cheat sheet and be able to use it more efficiently on the exam.