Draft of final Review for calculus 1A Spring 2009

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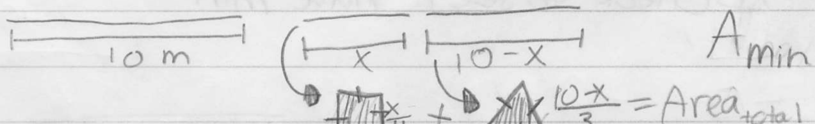
4.7) Optimization problems

1) draw a picture!
2) assign letters to unknowns
3) express what's given w/ letters
4) " " asked " "
5) identify what's asked
6) eliminate variables
7) formulate an equation & solve/max/min

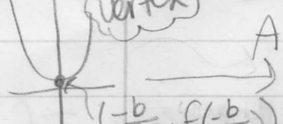
Examples:

Q - A piece of wire 10m long cut into 2 pieces one piece is bent into a square & the other into an equilateral Δ, max area enclosed?



A_{min} @ $x=?$ $\quad 0 \le x \le 10$

parabola min @ vertex



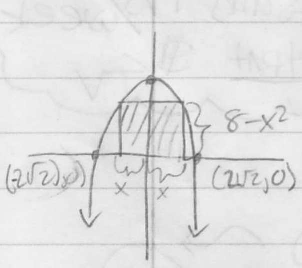
$$\text{Area} = \left(\frac{x}{4}\right)^2 + \frac{\sqrt{3}}{4}\left(\frac{10-x}{3}\right)^2 = \frac{x^2}{16} + \frac{\sqrt{3}}{4}\left(\frac{100-20x+x^2}{9}\right) = \frac{1}{4}\left(\frac{x^2}{4} + \frac{100\sqrt{3}}{9} - \frac{20\sqrt{3}x}{9} + \frac{\sqrt{3}x^2}{9}\right)$$

$$A = \frac{1}{4}\left[\left(\frac{1}{4}+\frac{\sqrt{3}}{9}\right)x^2 - \frac{20\sqrt{3}}{9}(x) + \frac{100\sqrt{3}}{9}\right]$$

$$x = \frac{-b}{2a} = \frac{20\sqrt{3}}{9}\left(\frac{\cancel{36}^{4}}{\cancel{2}(9+4\sqrt{3})}\right) = \boxed{\frac{40\sqrt{3}}{(9+4\sqrt{3})}} \Rightarrow \text{when } x = \frac{160\sqrt{3}}{9+4\sqrt{3}}$$

$A = \min$

Q - maximize area of rectangle w/ base on x-axis & 2 vertices on $y = 8-x^2$

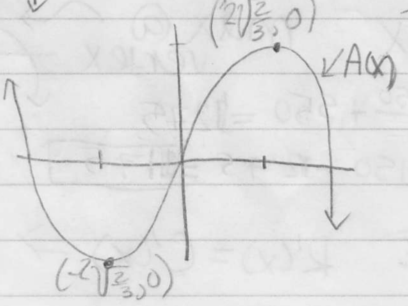


$\text{Area} = 2x(8-x^2) = 16x - 2x^3$ max?

$(\text{Area})' = 16 - 6x^2 = 0 = 2(8-3x^2)$

$= -3x^2 = -8$

$\sqrt{x^2} = \sqrt{\frac{8}{3}} \Rightarrow x = \pm 2\sqrt{\frac{2}{3}}$



max A @ $x = \boxed{2\sqrt{\frac{2}{3}}}$

$\boxed{\text{4.7 (cont.) App to Buisness \& Economics}}$

DEFENITIONS

marginal = derivative

- Cost of production $C(x)$ – cost to produce x-units
- marginal cost $= C'(x)$ = rate change of cost/Δ in units x produced
- avg cost $= \boxed{c(x) = C(x)/x}$
- price (demand) $p(x)$ = price/unit when you sell x-units
- Revenue $= R(x) = p(x)\,x$ = \$ from sale of x units (linear fn
- Profit $= \boxed{P(x) = C(x) - R(x) = C(x) - p(x)x}$
- marginal revenue $= R'(x)$ \$/unit
- marginal profit $= P'(x)$ \$/unit

Thm If avg cost, $c(x)$, is min ⇒ then avg cost = marginal cost $C'(x)$
:ie tan slope = secant line thru origin

⚠ check to see if have min

Thm If profit is maximal then $\boxed{R'(x) = C'(x)}$
⚠ check $R''(x_0) < C''(x_0)$

381-87 Examples Q: A manufacturer sells 1000 TVs/week @ \$450/TV. If he ↓ \$10 ⇒ ↑ sells 100/week

a) $p(x) = ?$

$p(1000) = \$450$ slope $= \frac{\Delta \$}{\Delta \text{sales}} = \frac{-10}{100} = -\frac{1}{10}$

$p(x)$ = linear fn
input = sales TVs/week
output \$/TV

$y - y_1 = m(x - x_1) \Rightarrow y - 450 = -\frac{1}{10}(x - 1000)$

Demand/price fn $= \boxed{p(x) = -\frac{x}{10} + 550}$

b) rebate to max $R(x)$

$R(x) = x[p(x)] = -\frac{x^2}{10} + 550x$ max @ vertex

$x = -550\left(\frac{10}{-2}\right) = 2750$ TVs $p(x) = -\frac{2750}{10} + 550 = \$275$

Rebate $= \$450 - \$275 = \boxed{\$175}$

c) $C(x) = 68000 + 150x$ rebate max profit = ? $R'(x) = C'(x) \Rightarrow$ max $P(x)$

$150 = -\frac{2x}{10} + 550$

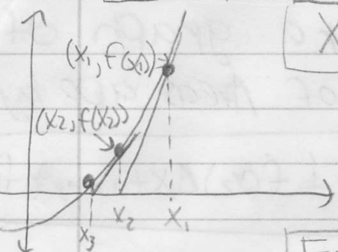
$-2000 = -x$

$x = 2000$ TVs

$p(x) = -\frac{2000}{10} + 550 = \$350/\text{TV}$

rebate $= \$450 - \$350 = \boxed{\$100}$

4.8) Newtons Method

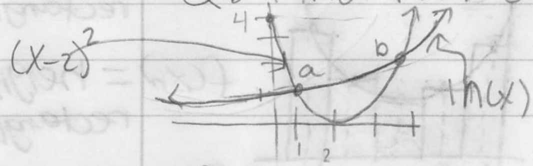


$$\boxed{X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)}}$$ for appx roots of fn's

- sketch graph if can & pick an x_1 close to root
- just keep iterating until #'s start repeating
- define $f(x)$ & $f'(x)$

Example

Q: find root of $(x-2)^2 = \ln x$ to 3 decimal places



$X_1 = 1 \quad X_2 = 1 - \frac{\ln(x) - (x-2)^2}{\frac{1}{x} - 2(x-2)} \rightarrow 1 - \frac{\ln(1) - (1-2)^2}{\frac{1}{1} - 2(1-2)} = \frac{4}{3}$

$X_3 \approx 1.4086$

$f(x) = \ln(x) - (x-2)^2$ $\quad X_4 \approx 1.4124$

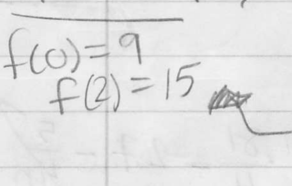
$f'(x) = \frac{1}{x} - 2(x-2)$ $\quad \boxed{X_5 \approx 1.4124}$

once amt #'s you want repeat your done

4.9 Antiderivatives

Indefinate integral no bounds $\int f(x)\,dx = F(x) + C \Rightarrow$ vary by a constant

Def A fn F is called an antiderivative of f on I if $F'(x) = f(x)$ for all $x \in I$ $\Rightarrow$ fn continuous on I



$\int \frac{1}{x}\,dx = \begin{cases} \ln|x| + C_1 & x > 0 \\ \ln|x| + C_2 & x < 0 \end{cases}$ 2 diff constants

! check continuity

example:

Q: Find $f(x)$ $\quad f''(x) = 2 - 12x \Rightarrow f'(x) = \int (2 - 12x)\,dx = 2x - 6x^2 + C_1$

$f(0) = 9$

$f(2) = 15$

$f(x) = \int (2x - 6x^2 + C_1)\,dx = x^2 - 3x^3 + C_1 x + C_2$

use given pt's to find C_1, C_2

$f(0) = 0^2 - 3(0)^3 + C_1(0) + C_2 = 9$

$\boxed{C_2 = 9}$

$f(2) = (2)^2 - 3(2)^3 + C_1(2) + 9 = 15$

$4 - 3(8) + 9 - 15 = (C_1)(-2)$

$-20 - 15 + 9 = -2C_1$

$-36 = -2C_1$

$\boxed{C_1 = 18}$

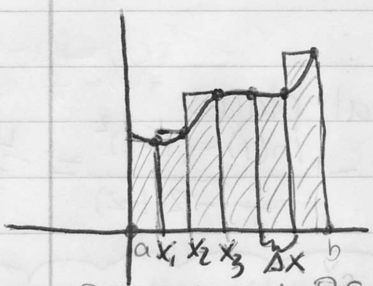
plug constants into equation

$$\boxed{f(x) = -3x^3 + x^2 + 18x + 9}$$

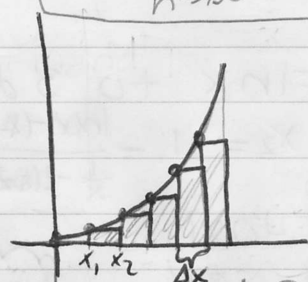
5.1) Riemann Sums (RS)

Def Area of a region, S, that lies under a graph of continuous function f = lim sum of Areas app by rectangles

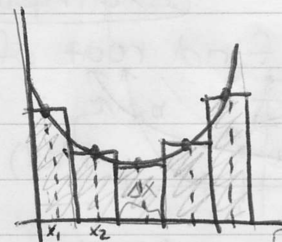
$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x\right]$$



Right hand RS



left hand RS



midpoint RS

Δx = width rectangle

$f(x_n)$ = height rectangle

Def Definate Integral

$$\text{Net Area} = \int_a^b f(x)\,dx = \lim_{x\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$\Delta x = \frac{b-a}{n}$$

$$x_i = a + \frac{i(b-a)}{n}$$

Example:

Q - Evaluate RS for $f(x) = x^3 - 6x$ right hand RS, $a=0$ $b=3$

$\Delta x = \frac{3-0}{n} = \frac{3}{n}$

$x_i = 0 + \frac{3i}{n}$

$$\int_0^3 (x^3-6x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right] = \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\frac{27i^3}{n^3} - \frac{18i}{n}$$

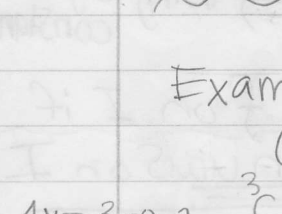
$$= \lim_{n\to\infty}\frac{3}{n}\left[\frac{27}{n^3}\sum_{i=1}^{n} i^3 - \frac{18}{n}\sum_{i=1}^{n} i\right] = \lim_{n\to\infty}\frac{3}{n}\left[\frac{27}{n^3}\left(\frac{n(n+1)}{2}\right)^2 - \frac{\overset{9}{\cancel{18}}}{\cancel{n}}\left(\frac{\cancel{n}(n+1)}{\cancel{2}}\right)\right]$$

You can bring "n" in front of RS but not lim

$$= \lim_{n\to\infty}\frac{3}{n}\left[\frac{27}{n^3}\left(\frac{n^2(n^2+2n+1)}{4}\right) - 9(n+1)\right]$$

$$= \lim_{n\to\infty}\frac{3}{n}(9)\left[\frac{3}{4}n + \frac{6}{4} + \frac{3}{4} - n - 1\right] = \lim_{n\to\infty}\frac{81}{4} - 27 - \frac{5}{4n}$$

$x(x^2-6)$



$$= \int_0^3 (x^3-6x)\,dx = \boxed{\frac{-27}{4}}$$

your left w/ constants & the rest go to Ø as $n\to\infty$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$\sum_{i=1}^{n} c = cn$$

5.3 Fundamental Thm of Calculus

- thm FTCI let $f(x)$ be cont on $[a,b]$ then
$$\boxed{\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)}$$

- thm FTC let $f(x)$ be cont on $[a,b]$ then $f(x)$ has an antiderivative $F(x)$ given by
$$\boxed{F(x) = \int_a^x f(t)\,d(t)} \quad \& \quad F'(x) = f(x)$$

Corellary

let $G(x) = \int_{h_1(x)}^{h_2(x)} f(t)\,dt$ then
$$\boxed{G'(x) = h_2'(x)\big(f(h_2(x))\big) - h_1'(x)\big(f(h_1(x))\big)}$$

Examples

Q1: $y = \int_{\cos x}^{5(x)} \cos(u^2)\,du \qquad y' = (5x)'\cos(5x^2) - (\cos x)'\cos(\cos^2 x)$

$$y' = \boxed{5\cos(25x^2) + (\sin x)[\cos(\cos^2 x)]}$$

Q2: $\int_{-2}^{2} f(x)\,dx$ where $f(x) = \begin{cases} 2, & -2 \le x \le 0 \\ 4 - x^2, & 0 < x \le 2 \end{cases}$

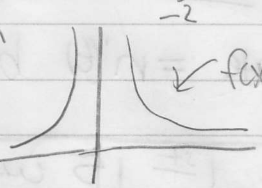
$$\int_{-2}^{2} f(x)\,dx = \int_{-2}^{0} 2\,dx + \int_0^2 (4 - x^2)\,dx = 2x\Big|_{-2}^{0} + 4x - \frac{x^3}{3}\Big|_0^2$$

can seperate into 2 intergals

$$= 0 + (+4) + 8 - \frac{8}{3} = \boxed{28/3}$$

Q3: $\int_{-2}^{1} x^{-4}\,dx \neq \frac{x^{-3}}{-3}\Big]_2^1 \neq -\frac{3}{8}$

check for cont. w/in interval



you neen a CONT interval to integrate

Q4: $\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \sqrt{\frac{3}{n}} + \dots + \sqrt{\frac{1}{n}}\right) = \int_a^b f(x)\,dx$

$$= \boxed{\int_0^1 \sqrt{x}\,dx}$$

$\Delta x = \frac{b-a}{n} = \frac{1}{n}$

$x_i = a + \frac{(b-a)i}{n}$

Identify interval $[a,b]$

" $f(x)$

use to find unknowns

$a = 0$

5.4) net Δ thm & indefinate $\int$'s

Total Δ thm: if $f(x)$ is cont on $[a,b]$, & $F(x)$ = some antiderivative then $F'(x) = f(x)$ &

by FTCI $\int_a^b f(x) = F(b) - F(a) \Rightarrow \boxed{\int_a^b F'(x) = F(b) - F(a)}$ total/net Δ of $F(x)$

Example:

$v(t) = 3t - 5$ m/hr $0 \le t \le 3$hr

Q1) what's dislacement $\int_0^3 (3t-5)\,dt = \frac{3}{2}t^2 - 5t \Big|_0^3 = \frac{27}{2} - \frac{30}{2} = \boxed{\frac{3}{2}m}$

Q2) whats total distance traveled $\int_0^3 |3t-5|\,dt = \int_0^{5/3} (5-3t)\,dt + \int_{5/3}^{3} (3t-5)\,dt$

$s(t) = |v(t)|$

$\int_a^b v(t)\,dt$ = displacement

&

$\int_a^b s(t)\,dt$ = total distance traveled

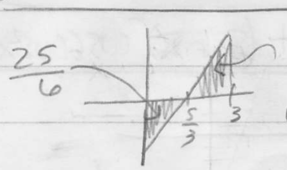
$= -\frac{3t^2}{2} + 5t \Big|_0^{5/3}$

$-\frac{3}{2}\left(\frac{5}{3}\right)^2 + \frac{25}{3} = \frac{25}{3}\left(-\frac{3}{6} + 1\right)$

$\frac{25}{6}$ m

$= \frac{3}{2}t^2 - 5t \Big|_{5/3}^{3}$

$\frac{27}{2} - 15 + \left(+\frac{25}{6}\right) = \frac{8}{3}$



area = total distance traveled

$D = \frac{25}{6} + \frac{8}{3} = \boxed{\frac{41}{6}m}$

indefinate $\int$ = family of fns

Q3) $\int \left(x^2 + 1 + \frac{1}{x^2+1}\right)dx = \int x^2\,dx + \int dx + \int \frac{1}{x^2+1} = \boxed{\frac{x^3}{3} + x + \arctan x + C}$

Q4) $\int_{-1}^{0} (2x - e^x)\,dx = x^2 - e^x \Big|_{-1}^{0} = (0^2 - e^0) - ((-1)^2 - (e^{-1})) = \boxed{-2 + \frac{1}{e}}$

(Definate $\int$ = finite #)

Q5) Honebee POP start w/ 100 bees & $\nearrow$ rate $= n'(t)$ bees/week

what's $100 + \int_0^{15} n'(t)\,dt$ represent

$\boxed{\text{total population of bees for 1st 15 weeks}}$

Q6) $f(x)$ = newtons x = meters what are units for $\int_0^{100} f(x)\,dx$

$\int_0^{100} f(x)$ = area under curve $A = m(\text{Newtons}) = \boxed{Nm}$ = units

5.5 Substitution

$$F(g(x))' \overset{CR}{=} F'(g(x))\, g'(x) = F'(u)\, u'$$

$$\int \underbrace{F'(g(x))}_{f(u)} \underbrace{g'(x)\, dx}_{du} = \int F'(u)\, du = F(u) + C = F(g(x)) + C$$

Q1) Example:

$u = 1+x^2$
$du = 2x\,dx$

$$\int \sqrt{1+x^2}\; x^5\, dx$$

if $u = 1+x^2 \Rightarrow x^2 = u-1$ substitute

$$\int u^{1/2}\, du\; x^4$$

$$\int u^{1/2}(u-1)^2\, du = \int u^{1/2}(u^2 - 2u + 1)\, du$$

$$= \int (u^{5/2} - 2u^{3/2} + u^{1/2})\, du = \frac{2}{7}u^{7/2} - 2\left(\frac{2}{5}\right)u^{5/2} + \frac{2}{3}u^{3/2} + C$$

put in terms of original variable

$$= \boxed{\frac{2}{7}(1+x^2)^{7/2} - \frac{4}{5}(1+x^2)^{5/2} + \frac{2}{3}(1+x^2)^{3/2} + C}$$

Q2) $\frac{1}{2}\int \left(\frac{2x}{1+x^4}\right) dx = \frac{1}{2}\int \frac{1}{1+u^2}\, du = \boxed{\frac{1}{2}\arctan(x^2)}$

$u = x^2$
$du = 2x\,dx$

Q3) $\int_0^{\pi/2} [\cos x \sin(\sin x)]\, dx = \int_0^{\pi/2} \sin u\, du = -\cos u \Big|_0^1$

$u = \sin x$
$du = \cos x\, dx$

(you can Δ bounds & evaluate in terms of u)

$= -\cos(0) - (-\cos(1))$

$= \boxed{-1 + \cos(1)}$

lower bound $= \sin(0) = 0$
upper bound $= \sin(\pi/2) = 1$

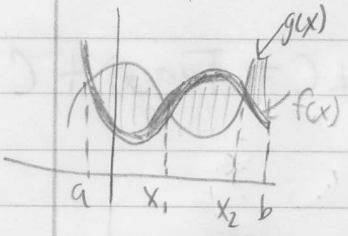
Q4) $\int \left(\frac{1+x}{1+x^2}\right) dx = \int \frac{1}{1+x^2}\, dx + \frac{1}{2}\int \frac{2x}{1+x^2}\, dx$ (seperate fraction to simplify Q)

$\boxed{\arctan x + \frac{1}{2}\ln|1+x^2| + C}$

$u = 1+x^2$
$du = 2x\,dx$

$\boxed{\text{6.1 Areas btw Curves}}$

Area btw curves $y=f(x)$ & $y=g(x)$ btw $x=a$ & $x=b$ is



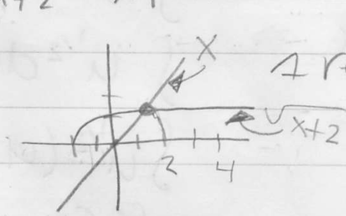
$$A = \int_a^b |f(x)-g(x)|\,dx \qquad A \geq 0 \text{ if neg } \underline{\text{wrong}}$$

$$A = \int_a^{x_1} (g(x)-f(x))dx + \int_{x_1}^{x_2} (f(x)-g(x))dx + \int_{x_2}^{c} (g(x)-f(x))dx$$

Example: (Q1) Evaluate $\int_0^4 |\sqrt{x+2}-x|\,dx$

1st figure out when intersect

$$\sqrt{x+2} = x$$
$$x+2 = x^2$$
$$= (x-2)(x+1)$$
$$x = 2$$



1 real solution

2nd divide into diff intervals & integrate

$$\int_0^4 |\sqrt{x+2}-x|\,dx = \int_0^2 (\sqrt{x+2}-x)\,dx + \int_2^4 (x-\sqrt{x+2})\,dx =$$

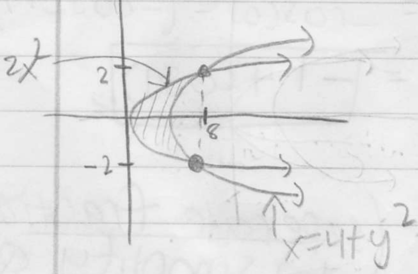
$$= \frac{2}{3}(x+2)^{3/2} - \frac{x^2}{2}\Big|_0^2 + \frac{x^2}{2} - \frac{2}{3}(x+2)^{3/2}\Big|_2^4$$

$$= \frac{2}{3}(4)^{3/2} - \frac{4}{2} - \left(\frac{2}{3}(2)^{3/2} - 0\right) + \frac{\cancel{16}}{\cancel{2}}8 - \frac{2}{3}(6)^{3/2} + \left(+\left(\frac{16}{3} - 2 - \frac{2}{3}\sqrt[3]{4}\right)\right)$$

$$\left(\frac{16}{3} - 2 - \frac{2}{3}\sqrt[3]{4}\right)2 + 8 - \frac{2}{3}\sqrt[3]{36}$$

$$= \frac{32}{3} - \cancel{4} - \frac{4}{3}\sqrt[3]{4} + \cancel{8}4 - \frac{2}{3}\sqrt[3]{36}$$

(Q2) Evaluate Area bounded by $x = 2y^2$ & $x = 4+y^2$

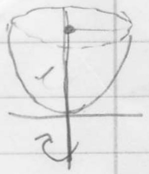


$$2y^2 = 4+y^2$$
$$y^2 = 4 \qquad x = 2(2)^2 = 8$$
$$y = \pm 2$$

$$A = \int_{-2}^{2} (4+y^2)-(2y^2)\,dy = \int_{-2}^{2} 4-y^2\,dy = 4y - \frac{y^3}{3}\Big|_{-2}^{2}$$

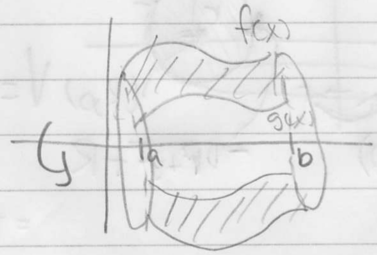
$$= 8 - \frac{8}{3} + \left(+8 - \frac{8}{3}\right) = 16 - \frac{16}{3} = \boxed{\frac{32}{3}}$$

6.2 Volumes



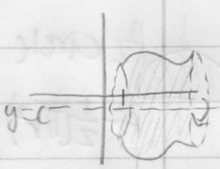
- thm let $f(x)$ be cont on $[a,b]$, $f(x) \geq 0$ $\forall x \in [a,b]$ then volume by revolving $y=f(x)$ about x-axis

$$\boxed{V = \int_a^b \pi f^2(x)\,dx}$$



- where $f(x) \geq g(x)$ $\forall x \in [a,b]$

$$\boxed{V = \int_a^b \pi\left(f^2(x) - g^2(x)\right)dx}$$



- if rotate $f(x)$ on $[a,b]$ about $y=c$ axis

$$\boxed{V = \int_a^b \pi\left((f(x)-c)^2\right)dx}$$

- if 2 fn's being rotated about $y=c$

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$f(x)$ = further from axis of rotation

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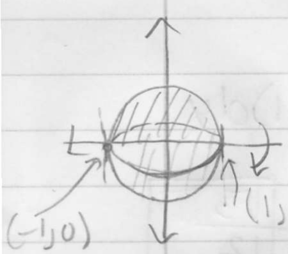
examples

Q1: $y = 1-x^2$ $y=0$ rotate region bounded by fn about x-axis

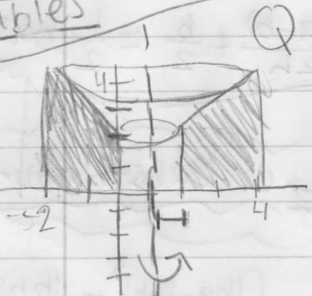


$$V = \pi\int_{-1}^{1}(1-x^2)^2\,dx = \pi\int_{-1}^{1}(1-2x^2+x^4)\,dx = \pi\left(x - \tfrac{2}{3}x^3 + \tfrac{x^5}{5}\right)\Big|_{-1}^{1}$$

even fn on an even interval doubles

$$2\pi\left(x - \tfrac{2}{3}x^3 + \tfrac{x^5}{5}\right)\Big|_0^1 = 2\pi\left(1 - \tfrac{2}{3} + \tfrac{1}{5}\right) = 2\pi\left(\tfrac{8}{3}\right) = \boxed{\frac{16\pi}{3}}$$

Q 2) $y=x$ $y=0$ $x=2$ $x=4$ about $x=1$



$$V = \pi\left[\int_0^4 (4-1)^2\,dy - \int_0^2 (2-1)^2\,dy - \int_2^4 (y-1)^2\,dy\right]$$

$$= \pi\left[\int_0^4 9\,dy - \int_0^2 1\,dy - \int_2^4 (y^2-2y+1)\,dy\right]$$

Since inner radius isn't constant you must seperate intergal into 2 diff intervals

axis of rotation vertical line = integrate in terms of y

$$= \pi\left(9y\Big|_0^4 - y\Big|_0^2 - \left(\tfrac{y^3}{3} - y^2 + y\Big|_2^4\right)\right)$$

$$= \pi\left(36 - 2 - \left(\tfrac{64}{3} - 16 + 4 - \left(\tfrac{8}{3} - \tfrac{4}{2} + 2\right)\right)\right)$$

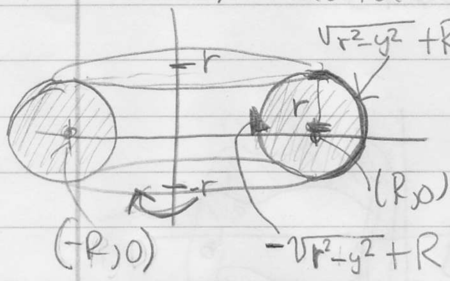
$$= \pi\left(34 - \tfrac{26}{3}\right) = \boxed{76\pi/3}$$

Integrate in terms of the axis the crossection is $\perp$ to

ie: if crossection $\perp$ to a vertical axis integrate in terms of y

" " " " horizontal " " " " of X

(Q3) find the volume of the taurus = a circle centered @ (R,0) w/ radius r rotated about x-axis

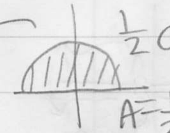


$$y^2+(x-R)^2=r^2 \Rightarrow x=\pm\sqrt{r^2-y^2}+R$$

$$V=\pi\int_{-r}^{r}\left(\sqrt{r^2-y^2}+R\right)^2-\left(-\sqrt{r^2-y^2}+R\right)^2dy$$

$$=\pi\int_{-r}^{r}\cancel{r^2}-\cancel{y^2}+2R\sqrt{r^2-y^2}+\cancel{R^2}-\left(\cancel{r^2}-\cancel{y^2}-2R\sqrt{r^2-y^2}+\cancel{R^2}\right)dy$$

$$=\pi\int_{-r}^{r}4R\sqrt{r^2-y^2}\,dy=4R\pi\int_{-r}^{r}\sqrt{r^2-y^2}$$

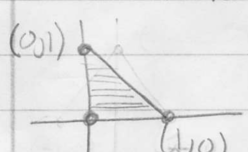


$\frac{1}{2}$ circle

$A=\frac{1}{2}(\pi r^2)$

$$=4R\pi\left(\frac{\pi r^2}{2}\right)=\boxed{2R\pi^2r^2}$$

(Q4) base of S is Δ region w/ vertices (0,0), (1,0), & (0,1) cross-section $\perp$ to y-axis are equilateral Δ



a) get equation of line

$$m=\frac{1}{-1}=-1 \quad y-0=-1(x-1)\Rightarrow x=-y+1$$

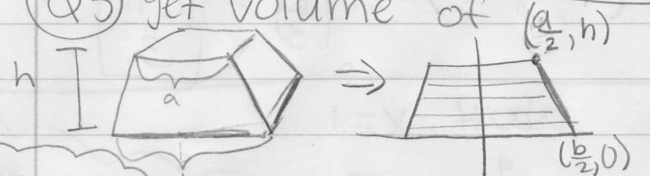
$$A_{equil}=\frac{\sqrt{3}}{4}s^2 \quad s=x=-y+1$$

Since crossection $\perp$ to y-axis integrate in terms of y

$$\frac{\sqrt{3}}{4}\int_0^1(-y+1)^2dy=\frac{\sqrt{3}}{4}\int_0^1(y^2-2y+1)dy$$

$$=\frac{\sqrt{3}}{4}\left(\frac{y^3}{3}-y^2+y\Big|_0^1\right)=\frac{\sqrt{3}}{4}\left(\frac{1}{3}\cancel{-1}\cancel{+1}\right)=\boxed{\frac{\sqrt{3}}{12}}$$

(Q5) get volume of



$$m=\frac{h}{\frac{a-b}{2}}=\frac{2h}{a-b} \quad y-0=\frac{2h}{a-b}\left(x-\frac{b}{2}\right)$$

$$x=\frac{y(a-b)}{2h}+\frac{b}{2}=\frac{1}{2}\text{ base of square}$$

Square top & bottom cross-sectional area = square

$$V=\int(\text{Side of }\square)^2=V=\int_0^{\frac{a}{2}}\left(2\left(\frac{y(a-b)+bh}{2h}\right)\right)^2dy$$

$$=\frac{1}{(a-b)2h^2}\int_0^{\frac{a}{2}}(y(a-b)+bh)^2(a-b)$$

$u=(a-b)y+bh$
$du=(a-b)dy$
= u substitution

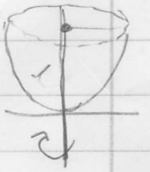
change boundary because it's in terms of y & need it for "u"

$$=\frac{1}{2h^2(a-b)}\int_{bh}^{ha-2hb}u^2dy=\frac{1}{(a-b)2h^2}\left(\frac{1}{3}u^3\Big|_{bh}^{ha-2hb}\right)=\frac{1}{6h^2(a-b)}\left[(ha-2hb)^3-(bh)^3\right]$$

$$V=\frac{h^3}{6h^2(a-b)}\left(a^3-3a^2(2b)+3a(2b)^2-8b^3+b^3\right)=\boxed{\frac{h}{6(a-b)}\left[a^3-6a^2b+12ab^2-7b^3\right]}$$

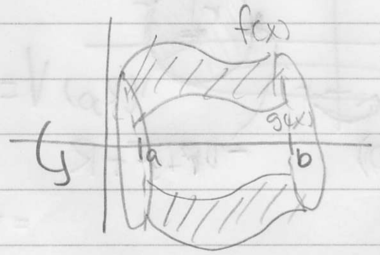
6.2 Volumes

- thm let $f(x)$ be cont on $[a,b]$, $f(x) \geq 0 \ \forall x \in [a,b]$ then volume by revolving $y=f(x)$ about x-axis



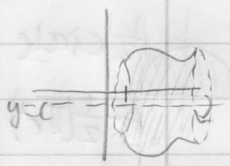
$$\boxed{V = \int_a^b \pi f^2(x)\,dx}$$

- where $f(x) \geq g(x) \quad \forall x \in [a,b]$



$$\boxed{V = \int_a^b \pi\left(f^2(x) - g^2(x)\right)dx}$$

- if rotate $f(x)$ on $[a,b]$ about $y=c$ axis



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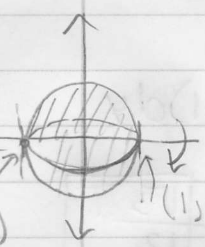

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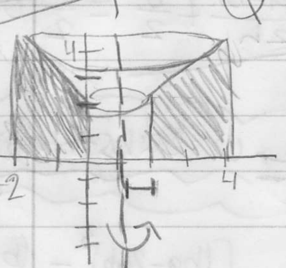

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$\boxed{6.5 \text{ MVT}}$

Def let $f(x)$ be cont on $[a,b]$ then its avg value is defined as

$$\boxed{f_{avg} = \frac{1}{(b-a)}\left[\int_a^b f(x)\,dx\right]} = \frac{\text{"}\int\text{"}}{[\ \ ]}\ \frac{\text{intergal}}{\text{interval}}$$

thm let $f(x)$ be cont. on $[a,b]$ then $f(x)$ attains somewhere on $[a,b]$ its avg value

$$f_{avg} = \frac{\int_a^b f(x)dx}{b-a} = f(c) \qquad c \in [a,b]$$

$$\boxed{\int_a^b f(x)\,dx = f(c)(b-a)}$$

Example:

Q. $f(x) = 2\sin x - \sin 2x$ on $[0,\pi]$

find f_{avg}

$$\frac{1}{\pi}\int_0^{\pi}(2\sin x - \sin 2x)dx = \frac{1}{\pi}\left(-2\cos x\Big|_0^{\pi} + \frac{\cos 2x}{2}\Big|_0^{\pi}\right)$$

$$= \frac{1}{\pi}\left(-2\cos(\pi) + 2\cos(0) - \left(\frac{\cancel{\cos(2\pi)}}{2} - \frac{\cancel{\cos 0}}{2}\right)\right)$$

$$= \boxed{4/\pi}$$

Q_2: $f(x) = 3x^2 + 5x$ on $[0, 3]$ find the avg

$$\frac{1}{3-0}\int_0^3 (3x^2+5x)dx = \frac{1}{3}\left(x^3 + \frac{5}{2}x^2\Big|_0^3\right)$$

$$= \frac{1}{3}\left(27 + \frac{5(9)}{2}\right) = \boxed{\frac{99}{6}}$$

when does fn attain f_{avg}?

$$3x^2 + 5x = \frac{\cancel{99}\,33}{\cancel{6}\,2}$$

fn is continuous so it will reach f_{avg}

$$6x^2 + 10x - 33 = 0$$

$$x = \frac{-10 \pm \sqrt{100 + 24(33)}}{12} = \frac{-10 \pm \sqrt{792}}{12} = \frac{-\cancel{10}^{5} \pm \cancel{6}^{3}\sqrt{22}}{\cancel{12}\,6}$$

$$\boxed{@\ x = \frac{-5 \pm 3\sqrt{22}}{6}}$$