# Review Topics for Final Exam in Calculus 1A

Instructor: Zvezdelina Stankova

## 1. Definitions

Be able to **write** precise definitions for any of the following concepts (where appropriate: both in words and in symbols), to **give** examples of each definition, to **prove** that these definitions are satisfied in specific examples. Wherever appropriate, be able to **graph** examples for each definition. What is

(1) a *Riemann sum* $R_n$, $L_n$, $M_n$, $Random_n$? How do we set up Riemann sums? What are $\Delta x$, $x_i$, $x_i^*$?
(2) the *sigma notation*? How do we put expressions into the sigma notation? How do we convert expressions in the sigma notation back into *expanded notation*?
(3) the *area under a curve*? How do we represent it in terms of limits of Riemann sums? What is an overestimate (underestimate) of an area? Can some areas be negative?
(4) the *definite integral* of $f(x)$ from $a$ to $b$ in terms of Riemann sums? In terms of areas? What is the geometric interpretation of the definite integral? (Be careful here!) Can a definite integral be negative?
(5) the *area function* $g(x)$ under a curve $f(x)$? What is the relation between the area function $g(x)$ and the given function $f(x)$?
(6) an *indefinite integral*? an *antiderivative*? What are the similarities and differences between definite and indefinite integrals? Which can be thought of as a "continuation" of the other? Why? Why similar notation for both? How many antiderivatives does a continuous function have?
(7) "$du = u'(x)dx$"? Is this a formula? What does it mean and where is it used?
(8) an *odd* function? an *even* function? Where can these properties be used? What do they simplify?
(9) the *area between two curves*? Can it be negative?
(10) a *solid of revolution*? What is the axis of rotation? How are the outer and inner radii used? What is a *cross-section* of a solid? What is a *volume* of a general solid (not necessarily a solid of revolution)?
(11) the *arithmetic mean* of several numbers? the *average* value of a function? Is this average value always attained by the function? (or should we say something about our function?)

## 2. Theorems

Be able to **write** what each of the following theorems (laws, propositions, corollaries, etc.) says. Be sure to understand, distinguish and **state** the conditions (hypothesis) of each theorem and its conclusion. Be prepared to **give** examples for each theorem, and most importantly, to **apply** each theorem appropriately in problems. The latter means: decide which theorem to use, check (in writing!) that all conditions of your theorem are satisfied in the problem in question, and then state (in writing!) the conclusion of the theorem using the specifics of your problem.

(1) **Properties of definite integrals**
   (a) *Algebraic properties*
      – addition and subtraction of functions: $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$;
      – multiplication by constants: $\int_a^b cf(x)dx = c\int_a^b f(x)dx$.
   (b) *Interval properties*
      – if bounds equal each other, then the integral is 0: $\int_a^a f(x)dx = 0$;
      – flip of bounds changes the sign of the integral: $\int_a^b f(x)dx = -\int_b^a f(x)dx$;
      – splitting of an interval into two (or more) intervals: $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.
   (c) *Comparison properties*
      – an integral of a positive function is positive: if $f(x) \geq 0$ for all $x \in [a, b]$ then $\int_a^b f(x)dx \geq 0$;
      – integrals respect inequalities of functions, i.e. an integral of a larger function is larger: if $f(x) \geq g(x)$ for all $x \in [a, b]$ then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$;

- sandwich property: if a function is sandwiched between two constant functions then its integral is also sandwiched between the integrals of the two constant functions, i.e. if $m \leq f(x) \leq M$ for all $x \in [a,b]$, then $\int_a^b m dx \leq \int_a^b f(x)dx \leq \int_a^b M dx$.

(d) *Basic Case*

- integral of a constant function is the area of the corresponding rectangle, i.e. $\int_a^b c dx = c(b-a)$.

(2) **The Fundamental Theorem of Calculus Part I:** the definite integral of $f(x)$ equals any antiderivative $F(x)$ of $f(x)$, evaluated at the ends of the interval, i.e. $\int_a^b f(x)dx = F(b) - F(a)$ where $F'(x) = f(x)$.

(3) **The Fundamental Theorem of Calculus Part II:** the area function $g(x) = \int_a^x f(t)dt$ has derivative equal to $f(x)$, i.e. $\frac{d}{dx}\int_a^x f(t)dt = f(x)$. Note the change of variables!

(4) **Summary of FTCI and II:** they show that differentiation and integration are inverse processes.

(5) **Formula for the derivative of more complex area functions**, in which one or both bounds of integration are "floating", i.e. given by functions instead of being constants:

$$\frac{d}{dx}\int_{h_1(x)}^{h_2(x)} f(t)dt = f(h_2(x)) \cdot h_2'(x) - f(h_1(x)) \cdot h_1'(x).$$

(6) **Table for Direct Integration:** must be committed to memory for efficient direct integration.

(7) **Total Change Theorem:** a rephrase of FTCII, i.e. $\int_a^b F'(x)dx = F(b) - F(a)$, and its corollaries in problems asking for total distance travelled, total displacement, etc.

(8) **Substitution Rule for indefinite integrals:** $\int f(g(x))g'(x)dx = \int f(u)du$ ($u = g(x)$, $du = u'(x)dx$.)

(9) **Substitution Rule for definite integrals:** $\int_a^b f(g(x))g'(x)dx = \int_{u(a)}^{u(b)} f(u)du$ ($u = g(x)$, $du = u'(x)dx$.)

(10) **Integrals of symmetric functions on symmetric intervals:**

(a) $\int_{-a}^a f(x)dx = 0$ if $f(x)$ is an odd function on $[-a,a]$, i.e. $f(-x) = -f(x)$ for all $x \in [-a,a]$.

(b) $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$ if $f(x)$ is even on $[-a,a]$, i.e. $f(-x) = f(x)$ for all $x \in [-a,a]$.

(11) **Areas between curves:**

(a) If $f(x) \geq g(x)$ on all of $[a,b]$ then the area between $f(x)$ and $g(x)$ is given by $\int_a^b (f(x) - g(x))dx$.

(b) If $f(x)$ and $g(x)$ change relative positions, i.e. somewhere $f(x)$ is the larger function, somewhere $g(x)$ is the larger function, then the area between $f(x)$ and $g(x)$ is given by $\int_a^b |f(x) - g(x)|dx$.

(12) **Volumes of Solids of Revolution**.

(a) If the function $f(x)$ on $[a,b]$ is rotated about the $x$-axis, then the volume is given by $\displaystyle\int_a^b \pi f^2(x)dx$.

If the function $f(y)$ is $[a,b]$ is rotated about the $y$-axis, then the volume is given by $\displaystyle\int_a^b \pi f^2(y)dy$.

In both cases, $a$ and $b$ are the endpoints of the solid **on the axis of revolution**.

(b) If a region between $f(x)$ and $g(x)$ is rotated about $y = c$, then the volume is given by

$$\int_a^b \pi\left((f(x)-c)^2 - (g(x)-c)^2\right)dx.$$

$a$ and $b$ are the endpoints of the solid **on the axis of revolution** $y = c$; $|f(x) - c|$ is the outer, and $|g(x) - c|$ is the inner radius of the "washer slice" (or the annulus cross-section of the solid.)

(c) If a region between $f(y)$ and $g(y)$ is rotated about $x = c$, then the volume is given by

$$\int_a^b \pi\left((f(y)-c)^2 - (g(y)-c)^2\right)dy.$$

$a$ and $b$ are the endpoints of the solid **on the axis of revolution** $x = c$; $|f(y) - c|$ is the outer, and $|g(y) - c|$ is the inner radius of the "washer slice".

(13) **Volumes of General Solids:**

(a) If $S(x)$ is the cross-section of a solid (with respect to planes perpendicular to the $x$-axis), then the volume of the solid is given by $\int_a^b S(x)dx$.
(b) Similarly, if the cross-sections are given by a function $S(y)$ of $y$, then the volume is $\int_a^b S(y)dy$.
(c) *Cavalieri's Principle:* If two solids have the same cross-sections with respect to a family of parallel planes, then the two solids have equal volumes.

(14) **Average Value of a Function:** If $f(x)$ is continuous on $[a, b]$, then the average value of $f(x)$ on $[a, b]$ is given by $f_{ave} = \dfrac{\int_a^b f(x)dx}{b-a}$. Be able to say this formula in words!

(15) **Mean Value Theorem for Integrals:** If $f(x)$ is continuous on $[a, b]$, then $f(x)$ attains its average value $f_{ave}$ for **at least one** $c \in [a, b]$: $f(c) = f_{ave}$ for some $c \in [a, b]$. Further, the integral of $f(x)$ can be represented as a rectangle with base $(b - a)$ and height $f(c) = f_{ave}$:

$$\int_a^b f(x)dx = f(c) \cdot (b - a).$$

## 3. Problem Solving Techniques

(1) **How do we set up Riemann Sums?** Let $f(x)$ be a continuous function on interval $[a, b]$. A "random" $n$-th approximation of $f(x)$ is the sum:

$$\text{Random}_n = (f(x_1^*) + f(x_2^*) + f(x_3^*) + \cdots + f(x_{n-1}^*) + f(x_n^*)) \cdot \Delta x,$$

where we partition the interval $[a, b]$ into $n$ small subintervals of length $\Delta = (b - a)/n$, and we choose sample points $x_1^*, ..., x_n^*$ in these $n$ small subintervals. Thus, for $n = 5$, let $x_i = a + i\dfrac{b-a}{5}$ for $i = 0, 1, 2, 3, 4, 5$; then the right, left and middle 5th approximations are:

$$R_5 = (f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)) \cdot \frac{(b-a)}{5} = \sum_{i=1}^{5} f(x_i)\Delta x,$$

$$L_5 = (f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4)) \cdot \frac{(b-a)}{5} = \sum_{i=0}^{4} f(x_i)\Delta x,$$

$$M_5 = \left(f\left(\frac{x_0 + x_1}{2}\right) + f\left(\frac{x_1 + x_2}{2}\right) + f\left(\frac{x_2 + x_3}{2}\right) + f\left(\frac{x_3 + x_4}{2}\right) + f\left(\frac{x_4 + x_5}{2}\right)\right) \cdot \frac{(b-a)}{5}.$$

(2) **How do we convert limits of Riemann sums into integrals?** The general formula is

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)dx.$$

Thus, $\lim_{n\to\infty} \sum_{i=1}^{n}$ is replaced by $\int_a^b$, $f(x_i^*)$ is replaced by $f(x)$, and $\Delta x$ is replaced by $dx$. The difficulty in these replacements usually occurs when trying to figure out what the interval $[a, b]$ is, and sometimes even which function exactly is being integrated.

(a) If you are fairly sure that you can identify the "sample points" $x_i^*$, then $a = \lim_{n\to\infty} x_1^*$, and $b = \lim_{n\to\infty} x_n^*$.
(b) You can also try to identify $\Delta x$. Make sure that it is of the form $\frac{constant}{n}$. The constant will then tell you the width of the total interval $[a, b]$.
(c) As an example, consider

$$\lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^9 + \left(\frac{2}{n}\right)^9 + \left(\frac{3}{n}\right)^9 + \cdots + \left(\frac{n}{n}\right)^9\right].$$

Clearly, they are telling us that $\Delta x = 1/n$. So, $b - a = 1$. Further, the sample points are $x_1^* = 1/n$, $x_2^* = 2/n$,..., $x_n^* = n/n$. Thus, $a = \lim_{n\to\infty} 1/n = 0$, and $b = \lim_{n\to\infty} n/n = 1$. The function is $f(x) = x^9$, and we can rewrite the above limit as:

$$\int_0^1 x^9 dx = \frac{x^{10}}{10}\Big|_0^1 = \frac{1}{10}.$$

(3) **How do we find derivatives of area functions?** You use FTC I, which states that

$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

If you are being asked a more complicated question where the bounds of integration ($a$ and $x$ above) are replaced by some other functions, then you apply directly Formula 5 from the Theorems section of this review handout.

(4) **How do we use the Total Change Theorem?** Usually, the relevant problem will deal with two functions: the "derivative" function $f(x)$, and the "antiderivative" (or integral) function $F(x)$: $F'(x) = f(x)$. It is your job to recognize which function is which. A hint here are the units in which the functions are given. For example, if one function is given in units pounds/years, and the other is given in pounds, then the first function, call it $f(x)$, is the derivative of the second function, call it $F(x)$: $F(x)$ is the weight in pounds at time $x$ years, and $f(x)$ is the rate of change of weight in pounds per year. The variable $x$ with respect to $F(x)$ is differentiated is time in years. Thus,

$$\int_a^b f(x)dx = F(b) - F(a)$$

can be interpreted as the **total change in weight from year $a$ to year $b$.**

A common class of problems gives the velocity function $v(t)$ on some interval $[a, b]$, and asks for the final displacement of the body at time $b$, or for the total distance travelled between time $a$ and $b$. The displacement function $F(t)$ is an antiderivative of $v(t)$, hence the final displacement at time $b$ with respect to time $a$ is given by:

$$\int_a^b v(t)dt = F(b) - F(a).$$

The speed is given by $|v(t)|$, and the total distance $D(t)$ travelled during the interval $[a, b]$ is given by:

$$\int_a^b |v(t)|dt = D(b) - D(a).$$

To calculate this last integral, you have to draw a very good graph of $v(t)$, flip all negative parts of the graph above the $x$-axis, and then calculate and add all individual areas under the graph of $|v(t)|$: be aware that **all** of these areas will be taken with **positive** signs in determining the total distance travelled, while some will be taken with negative signs when finding the final displacement.

(5) **How do we show Inequalities of Integrals?** If possible, draw a good graph of the function(s).

(a) If you are asked to show that $\int_a^b f(x)dx \geq 0$, you may check to see if $f(x) \geq 0$ on **all of** $[a, b]$. If yes, then the definite integral will equal the area under $f(x)$, and hence $\int_a^b f(x)dx \geq 0$. No calculations are necessary here except for verifying that $f(x) \geq 0$ for all $x \in [a, b]$.

(b) If are asked to show that $\int_a^b f(x)dx \geq \int_a^b g(x)dx$, you may check to see if $f(x) \geq g(x)$ on **all of** $[a, b]$. If yes, then the definite integral of $f(x)$ will be greater than the definite integral of $g(x)$ by a comparison property of integrals. No calculations are necessary here except for verifying that $f(x) \geq g(x)$ for all $x \in [a, b]$. For this, you may have to reason backwards: start with $f(x) \geq g(x)$ and step by step show that this inequality is the same as some obvious inequality; for example, if you have to show that $\sqrt{5-x} \geq \sqrt{x+1}$ on $[1, 2]$, you can reduce this inequality eventually to $6 \geq 2x$, or $3 \geq x$, which is obviously true for all $x \in [1, 2]$.

(c) If you are asked to show that $\int_a^b f(x)dx \geq B$ or $\int_a^b f(x)dx \leq A$, or both, try finding the maximum $M$ and the minimum $m$ of $f(x)$ on $[a, b]$. Then $m \leq f(x) \leq M$ for all $x \in [a, b]$, so by a comparison property of integrals, $\int_a^b m dx \leq \int_a^b f(x)dx \leq \int_a^b M dx$. Thus, compute $m(b-a)$ and $M(b-a)$ and hope that they equal $B$ and $A$, respectively.

(d) If the sandwich by constants in (c) doesn't work, then you need to do some more innovative thinking, e.g. increasing sin$x$ to 1, dropping part of the function to decrease it, replacing numerator or denominator by 1, etc.

(6) **How do we integrate discontinuous functions?** One has to be very careful when integrating discontinuous functions, since most of our theorems and formulas either fail completely for such functions, or need some modifications.

(a) If $f(x)$ has an **infinite discontinuity** on the interval $[a,b]$ (e.g. $f(x) = 1/(x-2)$ on $[1,5]$), then do **NOT** attempt to find $\int_a^b f(x)dx$! FTC fails completely here, and a new theorem is necessary. (To be studied later in 7.8. Improper Integrals.) If they are asking you for the indefinite integral, find an antiderivative for each subinterval on which $f(x)$ is continuous, and make sure you add different constants ($C_1$, $C_2$, etc.) for each such subinterval.

(b) If $f(x)$ has a **jump discontinuity** or a **removable discontinuity**, or is given by several different formulas, then partition the interval $[a,b]$ into several subintervals on each of which $f(x)$ is continuous (or is given by a single formula), and apply FTCII to each small subinterval **separately**. If they are asking you just for the indefinite integral of $f(x)$, make sure that you find the antiderivatives on each separate small subinterval and add a different constant for each subinterval.

(7) **How do we guess $u(x)$ and $u'(x)$ in the substitution rule?** There isn't any "bullet-proof" method for guessing (that's why it's called "guessing" afterall), but here are several tips that will suffice in all problems of Calculus II. By applying the Substitution Rule we are making an attempt to reverse the effect of differentiation by the Chain Rule. Thus, we want to represent our function as if it were the result of a Chain Rule: $f(g(x)) \cdot g'(x)$, where $u = g(x) =$ blah; in other words, we are trying to write our function as a product of a "$u$-part" and a "$u'(x)dx$-part". The key to "correct guessing" is to remember that, whatever $u(x)$ turns out to be, the derivative $u'(x)$ **must appear multiplied by** $dx$!

(a) Try to locate $u'(x)$: all candidates for $u'(x)$ are among the things with which $dx$ is multiplied.

(b) For example, $\int 3x^2 \cos(x^3)dx$ yields two possibilities for $u'(x)$: $u'(x) = 3x^2$ and $u'(x) = \cos(x^3)$. Now let's recall that we need to determine also $u(x)$, i.e. we have to do direct integration on our chosen $u'(x)$. Obviously, the choice $u'(x) = \cos(x^3)$ is bad because we don't know how to directly integrate $\cos(x^3)$. However, the choice $u'(x) = 3x^2$ works very well, since we can integrate immediately: $u(x) = x^3 + C$, and notice that $x^3$ appears in the rest of the expression. Thus, we set $u(x) = x^3$ and $du = (x^3)'dx = 3x^2dx$, and the substitution rule yields:

$$\int 3x^2 \cos(x^3)dx = \int \cos(x^3)(3x^2dx) = \int \cos(u)du = \sin(u) + C = \sin(x^3) + C.$$

Don't forget to check the answer by differentiation!!

(c) Recall that "denominators" may also yield something relevant for $u'(x)$. For example, in $\int \frac{\arctan x}{\sqrt{1+x^2}}dx$ it will be unsuitable to set $u'(x) = \arctan(x)$ (why?), but it will be very suitable to set $u'(x) = \frac{1}{\sqrt{1+x^2}}$: then $u(x) = \int \frac{1}{\sqrt{1+x^2}}dx = \arctan(x)$ and $du = \frac{1}{\sqrt{1+x^2}}dx$ so that

$$\int \frac{\arctan x}{\sqrt{1+x^2}}dx = \int \arctan x \cdot \left(\frac{1}{\sqrt{1+x^2}}dx\right) = \int u du = \frac{u^2}{2} + C = \frac{\arctan^2(x)}{2} + C.$$

Don't forget to check the answer by differentiation!!

(d) Sometimes we have readjust $u'(x)$ and $u(x)$ by constants. For example, in $\int x^2 \cos(x^3-7)dx$ we can initially guess $u'(x) = x^2$ so that $u(x) = \frac{x^3}{3} + C$. But this $u(x)$ does **not** conveniently appear in the rest of the function! Instead, we would have liked that $u(x) = x^3 - 7$: this is a harmless little wish, which can be satisfied on the spot by setting $du = (x^3-7)'dx = 3x^2dx$.

We can then "solve" $du = 3x^2dx$: $x^2dx = \frac{1}{3}du$, and substitute in the integral:

$$\int x^2\cos(x^3-7)dx = \int \frac{1}{3}\cos(u)du = \frac{1}{3}\sin(u)+C = \frac{1}{3}\sin(x^3-7)+C.$$

Don't forget to check the answer by differentiation!!

(e) We learn from the above that after having made a choice for $u'(x)$, and hence finding $u(x)$, we can readjust $u(x)$ by replacing it by any convenient expression of the form $C_1u(x)+C_2$ where $C_1$ and $C_2$ are some suitable constants. This forces a slight change in $u'(x)$: multiplication by the constant $C_1$. To summarize: always keep in mind that certain degrees of freedom in our choices for $u(x)$ and $u'(x)$ exist: for $u(x)$ there are "2 degrees of freedom", and for $u'(x)$ there is only "1 degree of freedom". Be aware that among all possible choices for $u(x)$ and $u'(x)$ there is one most convenient choice: it is dictated by what we would like $u(x)$ to be in our original function.

(f) Sometimes it may appear as if there are **no** candidates for the role of $u'(x)$. For example, in $\int e^{6x}dx$, $dx$ is multiplied only by $e^{6x}$, so what can $u'(x)$ be? It can be a constant: it can be any constant that we like. So, we look for a possible $u(x)$: it looks like $u=6x$ would be nice; hence $u'(x)=6dx$, $dx=\frac{1}{6}du$, and

$$\int e^{6x}dx = \int \frac{1}{6}e^u du = \frac{1}{6}e^u+C = \frac{1}{6}e^{6x}+C.$$

Don't forget to check the answer by differentiation!!

(g) Finally, some problems are trickier than we would have liked. The "trouble" there, surprisingly, is not in the guessing of $u(x)$ and $u'(x)$, but rather in trying to get rid of all $x$'s and replacing them by $u$'s. For example, in $\int \sqrt{x^2+1}x^5dx$, it is clear that we definitely want to get rid of $x^2+1$ under the radical, so we substitute $u(x)=x^2+1$, $du=2xdx$, but then how do we get rid of the remaining $x^4$ in our function? We solve for $x^4$ from our substitution equation: $u=x^2+1 \;\Rightarrow\; x^2=u-1 \;\Rightarrow\; x^4=(u-1)^2$, so that our integral will look like:

$$\int \sqrt{x^2+1}x^5dx = \int \sqrt{x^2+1}x^4xdx = \int \sqrt{u}(u-1)^2\cdot\frac{du}{2} = \int \frac{1}{2}\sqrt{u}(u^2-2u+1)du = \frac{1}{2}\int (u^{5/2}-2u^{3/2}+u^{1/2})du$$

Finish this example as we did in class, and check with differentiation.

(h) Continuing with the "mean streak" of substitution rule problems: splitting the integral as a sum of two integrals; also, don't forget that functions maybe even/odd, thus simplifying integration on symmetric intervals $[-a,a]$.

(8) **How do we set up integrals for Areas between Curves $f(x)$ and $g(x)$?** The problem will sometimes specify vertical bounds like $x=-2$ and $x=5$, or horizontal bounds like $y=-9$ and $y=-6$, or no other bounds when the region is completely determined by the two curves.

(a) Draw the given region and decide if you would like to integrate with respect to $x$ or $y$. The better choice is determined by the fewer subregions that you have to integrate: if you need to chop up your interval into many pieces and use many different functions on these pieces, probably this is not a good choice for an integration variable.

(b) Make sure that your curves are written in the same variable that you will be integrating with respect to: $f(x)$ and $g(x)$, or $f(y)$ and $g(y)$.

(c) Find the points of intersection of the two curves by setting them to equal each other: $f(x)=g(x)$ (or $f(y)=g(y)$).

(d) Determine the bounds of integration on the corresponding axis of integration. Determine the subintervals over each of which only one function is bigger than the other.

(e) Find the integral $\int_a^b |f(x)-g(x)|dx$ (or the corresponding $y$-integral). To calculate this integral, use your intervals in part (d) and sum up all integrals depending on which function is bigger/smaller in each subinterval. Thus, you may end up with something like

$$\int_a^c (f(x) - g(x))dx + \int_c^d (g(x) - f(x))dx + \int_d^b (f(x) - g(x))dx$$

if $f(x) \geq g(x)$ on $[a, c]$, $g(x) \geq f(x)$ on $[c, d]$, and $f(x) \geq g(x)$ on $[d, b]$. The key point is to sum up geometric areas - i.e. all areas are included with positive signs even though some areas maybe "under the $x$-axis." You must distinguish between the area between two curves and the definite integral of the difference of the two functions: $\int_a^b |f(x) - g(x)|dx$ is quite often different from $\int_a^b (f(x) - g(x))dx$ (Why?)

(f) If you have trouble drawing the graph of $x = f(y)$, switch for a moment the two variables: $y = f(x)$, draw the graph of $y = f(x)$, and then flip it across the line $y = x$. This transforms, for example, all $x$-intercepts of $y = f(x)$ into the $y$-intercepts of $x = f(y)$, etc, and it does transform the whole graph of $y = f(x)$ into the graph of $x = f(y)$.

(9) **How do we set up integrals to calculate Volumes of Solids of Revolution?**

(a) Decide with respect to which variable you will be integrating: this depends solely on what kind of axis you are rotating about. If it is a horizontal axis (e.g. $x$-axis or $y = c$) then you will be integrating wrt variable $x$ . If it is a vertical axis (e.g. $y$-axis or $x = c$) then you will be integrating wrt variable $y$.

(b) Find the beginning and the ending points of your solid measured on the axis you chose above. These will be your bounds of integration. Make sure you draw a very good picture!

(c) Make sure your functions are written in the "correct" integration variable. If not, rewrite the functions in this variable. For example, if you are going to integrate wrt variable $y$, but one function is written as $y = 2x^3 + 4$, then solve for $x$: $x = ((y - 4)/2)^{1/3}$.

(d) Next decide what your outer radius will be, and what your inner radius will be. Say, you are rotating a region between $f(y)$ and $g(y)$ about $x = -3$, and say, $f(y)$ is "further away" from $x = -3$ compared to $g(y)$. Then your outer radius will be $|(f(y) - (-3)|$ and your inner radius will be $|g(y) - (-3)|$: you subtract your axis of rotation from your two functions.

(e) Finally, apply the most general formula (the last formula we derived in class) about volumes of solids of revolution. In the example above, rotating about $y = -3$ produces

$$\int_a^b \pi \left((f(y) + 3)^2 - (g(y) + 3)^2\right) dy.$$

Note that here "+3" resulted from subtracting $(-3)$ from the two functions. If we had to rotate about $y = 4$, then the outer and innner radius would have been $|f(y) - 4|$ and $|g(y) - 4|$, respectively.

If you have only one function to rotate, there is no need to look for "outer" or "inner" radii. You will have only one radius generated by this function and the axis about which you are rotating.

(10) **How do we find Volumes of General Solids?** This is a hard question to answer because solids can come in many different shapes. First we choose a "convenient" axis with respect to which the integration will occur. The "convenience" is determined by an easy to find cross-section function $S(x)$. Next we find the beginning and ending points $a$ and $b$ of the solid as measured on our axis, and we integrate $\int_a^b S(x)dx$. Cavalieri's Principle is sometimes helpful here.

(11) **How do we apply Mean Value Theorem for Integrals in Problems?** Usually, such problems ask you something either about $\int_a^b f(x)dx$, or about a specific value of $f(x)$. So, first check that $f(x)$ is **contunuous** on $[a, b]$. Next, calculate the average value of $f(x)$ on $[a, b]$ using the formula

$$f_{ave} = \frac{\int_a^b f(x)dx}{b - a}.$$

Depending on what the problem is asking,

(a) conclude that $\int_a^b f(x)dx$ equals $f_{ave} \cdot (b - a)$.

(b) conclude that $f(x)$ attains its average value $f_{ave}$ for at least one value $c \in [a, b]$: $f(c) = f_{ave}$.
(c) set $f(c) = f_{ave}$ and solve for $c$: find **all** $c \in [a, b]$ at which $f(x)$ attains its average value $f_{ave}$.

## 4. Useful Formulas and Miscellaneous Facts

(1) The formula describing a circle of radius $r$ as a function: $y = \pm\sqrt{r^2 - x^2}$ or $x = \pm\sqrt{r^2 - y^2}$.
(2) Quadratic formula for solving quadratic equations.
(3) Formulas for area of a circle and the volume of a sphere of radii $r$: $\pi r^2$ and $\frac{4}{3}\pi r^3$, respectively.
(4) Fraction manipulations, exponential and logarithmic manipulations and formulas. Manipulations with the sigma notation: going back and forth between sigma notation and expanded notation.
(5) Formulas for specific sums:
(a) $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$. (b) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.
(6) Differentiation Laws, especially the Chain Rule and the Product Rule.

## 5. Cheat Sheet and Studying for the Exam

For the exam, you are allowed to have a "cheat sheet" - *one page* of a regular $8 \times 11$ sheet. You can write whatever you wish there, under the following conditions:

- The whole cheat sheet must be **handwritten by your own hand**! No xeroxing, no copying, (and for that matter, no tearing pages from the textbook and pasting them onto your cheat sheet.)
- Any violation of these rules will disqualify your cheat sheet and may end in your own disqualification from the exam. I may decide to randomly check your cheat sheets, so let's play it fair and square. :)
- Don't be a **freakasaurus**! Start studying for the exam several days in advance, and prepare your cheat sheet at least 2 days in advance. This will give you enough time to become familiar with your cheat sheet and be able to use it more efficiently on the exam.
- **Do NOT overstudy on the day of the exam!! More than 3 hours of math study on the day of the Final is counterproductive! No kidding!**