Review Topics for Midterm II in Calculus 1A

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1. Definitions

Be able to **write** precise definitions for any of the following concepts (where appropriate: both in words and in symbols), to **give** examples of each definition, to **prove** that these definitions are satisfied in specific examples. Wherever appropriate, be able to **graph** examples for each definition. What is

(1) *implicit* differentiation?
(2) a *global minimum* and a *global maximum* of a function?
(3) a *local minimum* and a *local maximum* of a function?
(4) a *global extremum* and a *local extremum* of a function? Is a local extremum necessarily a global extremum? examples? Is a global extremum necessarily a local extremum? examples? Can an endpoint be a local extremum? a global extremum?
(5) a *critical point*? How do we find all of them? What is a *potential* and a *realized* local extremum?
(6) a *nice* function?
(7) a *concave up* (*concave down*) function?
(8) an *inflection* point? How do we locate all of them?
(9) an *indeterminate* form? How do we deal with them?
(10) the method of *contradiction*? How do we use it?
(11) a *slant asymptote*? How do we find it?
(12) an *optimization problem*? What is the strategy for solving optimization problems?
(13) the *area* and *perimeter* of a figure? Basic examples of figures.

2. Theorems

Be able to **write** what each of the following theorems (laws, propositions, corollaries, etc.) says. Be sure to understand, distinguish and **state** the conditions (hypothesis) of each theorem and its conclusion. Be prepared to **give** examples for each theorem, and most importantly, to **apply** each theorem appropriately in problems. The latter means: decide which theorem to use, check (in writing!) that all conditions of your theorem are satisfied in the problem in question, and then state (in writing!) the conclusion of the theorem using the specifics of your problem.

(1) **Extreme Value Theorem.** To what functions is it applicable?
(2) **Theorem for Local Extrema.** If $f(x_0)$ is a local min/max and $f'(x_0)$ exists, then $f'(x_0) = 0$.
(3) **Contrapositive Statement.** If $f'(x_0) \neq 0$, then $f(x)$ cannot have a local extremum at x_0.
(4) **Converse Statement is False.** If $f'(x_0) = 0$, this does not guarantee that $f(x)$ has a local extremum at x_0. Why? Counterexample?
(5) **Rolle's Theorem.** To what functions is it applicable?
(6) **Mean Value Theorem.** To what functions is it applicable?
(7) **Cor.1.** If $f'(x) = 0\ \forall x$, then $f(x)$ is a constant function.
(8) **Cor.2.** If $f'(x) = g'(x)\ \forall x$, then $f(x)$ and $g(x)$ differ by a constant: $f(x) = g(x) + c$ everywhere.
(9) **Increasing/Decreasing Test.** Does it always work? Examples.
(10) **First Derivative Test.** Does it always work? Examples.
(11) **Second Derivative Test.** Does it always work? Examples.
(12) **Concavity Test.** Does it always work? Examples.
(13) **Inflection Point Test.** Does it always work? Examples.
(14) **L'Hospital's Rule.** Does it always work? Examples.
(15) **Distance Formula.** Pythagorus Theorem.
(16) **Area and Perimeter Formulas** for basic figures.

3. Problem Solving Techniques

(1) **How do we use implicit differentiation?** This is used to find tangent lines and their slopes to *curves* in the plane which are *not* graphs of functions (i.e. they violate the vertical line test). Thus, we don't have a function formula to differentiate, but instead an equation for the curve, e.g. $x^3+x^2y+4y^2=6$. It will be hard, sometimes impossible, to solve such an equation for y, and hence a formula for y may not be available. [1]

(a) We imagine that y is given by such a formula $y=f(x)$ (e.g. $x^3+x^2f(x)+4f(x)^2=6$), and we differentiate (with respect to x) both sides of the given equation, e.g.

$$(x^3+x^2y+4y^2)'=(6)' \;\Rightarrow\; 3x^2+2xy+x^2y'+8y\cdot y'=0$$

Do *not* forget to include y' wherever appropriate, for $y=f(x)$ so that $y'\neq 1$, but $y'=\frac{dy}{dx}=f'(x)$.

(b) Solve the above for y':

$$3x^2+2xy+y'(x^2+8y)=0 \;\Rightarrow\; y'=-\frac{3x^2+2xy}{x^2+8y}.$$

This is the best we can do for y': we have expressed it in terms of x and the original function y.

(c) If we are asked something about derivatives, slopes and tangents at specific places, then we use the above formula for y' and if necessary, the original equation for y. E.g. in our example, find the slope and the equation for the tangent at point $(1,1)$. [2] Now we use the formula for $y'(x)$ and substitute $x=1$, $y=1$:

$$y'(1)=-\frac{3+2}{1+8}=-\frac{5}{9}.$$

Finally, we use the point-slope formula:

$$y'(1)=\frac{y-y(1)}{x-1} \;\Rightarrow\; -\frac{5}{9}=\frac{y-1}{x-1} \;\Rightarrow\; y=-\frac{5}{9}x+\frac{14}{9}.$$

It is always good to check if this is the correct equation for the tangent line: yes, because the slope is $-5/9$, and if we plug in the point $(1,1)$ it works: $1=1$.

(d) Say, we want to find all points on the curve where the tangent to the curve is horizontal. In general, this is not an easy question to answer. Set $y'(x)=0$, and obtain two equations in terms of x and y: the derivative equation and the original equation. Now you are supposed to solve this system of two equations for x and y. In our example, this amounts to:

$$\left|\begin{array}{l} 0=-\frac{3x^2+2xy}{x^2+8y} \\ x^3+x^2y+4y^2=6 \end{array}\right.$$

The first equation yields $0=3x^2+2xy=x(3x+2y)$, i.e. $x=0$ or $y=-\frac{3}{2}x$. Substituting in the second equation: $4y^2=6$ (when $x=0$), or $x^3-\frac{3}{2}x^3+4\frac{9}{4}x^2=6$ (when $y=-\frac{3}{2}x$), i.e. $y=\pm\sqrt{\frac{3}{2}}$, while the second equation is a pain and I won't solve it here. The final answer would have been: the tangent lines to the curve are horizontal at points $(0,\sqrt{\frac{3}{2}})$, $(0,-\sqrt{\frac{3}{2}})$, and at the points yielded by the second case above.

The good news is that if a similar question appears on the exam, the calculations will be easier. The method, however, is outlined above. Note that similarly you can solve all sorts of questions about the tangents to such curves: e.g. find where the tangents are parallel to $y=x$ (set $y'(x)=1$), etc.

[1] In the particular example, you can indeed solve for y viewing the given equation as a quadratic equation in "y", but believe me, you probably don't want to do that, and you should follow the method of implicit differentiation instead.

[2] Note that this point was obtained by substituting $x=1$ into the original equation: $1+y+4y^2=6$ and obtaining solutions $y=1,-5/4$. Thus, you could have been asked to find instead the tangent line at point $(1,-5/4)$.

(2) **How do we apply Rolle's Thm in problems?** Rolle's Thm is used typically in problems of the type: *Show that a nice function $f(x)$ has at most k roots.*

Argue by contradiction. Start by supposing that $f(x)$ has more than k roots, i.e. that $f(x)$ has at least $k+1$ roots. Then between every two consecutive roots, by Rolle's Thm, the derivative function $f'(x)$ must have at least one root. Thus, $k+1$ roots of $f(x)$ mean that $f'(x)$ has at least k roots on its own. Study $f'(x)$ and show that it does **not** have these many roots. (For this, you may need to go down to $f''(x)$, or even higher derivatives, if necessary; but usually, a check of $f'(x)$ is sufficient.)

So, on the one hand, $f'(x)$ must have k roots, but on the other hand, it doesn't have k roots - well, that's a contradiction. Conclude that the supposition was wrong to start with. Finally, conclude that therefore $f(x)$ does indeed have at most k roots. $\square$

(3) **How do we apply MVT and its Corollaries in problems?**

(a) *Show that some identity is true.* First, label the two sides of the identity by $f(x)$ and $g(x)$. Find the two derivatives $f'(x)$ and $g'(x)$ and show that they are equal everywhere: $f'(x) = g'(x)\ \forall x$. Conclude by Cor.2 that $f(x) = g(x) + c$ for some constant c. Finally, substitute your favorite $x = a$ and show that $f(a) = g(a)$ (well, make sure that you choose your a so that you can easily calculate $f(a)$ and $g(a)$.) But from the above, $f(a) = g(a) + c$, so that $c = 0$. Conclude that $f(x) = g(x)$ everywhere. $\square$

(b) *Given some data about a function $f(x)$, show some inequality involving $f(x)$ and or $f'(x)$.* There are various problems of this type, and we can't describe in detail all of them. The general principle is as follows. Set up the MVT for $f(x)$ on convenient interval $[A, B]$:

$$f'(x_0) = \frac{f(B) - f(A)}{B - A} \text{ for some } x_0 \in (A, B).$$

Plug in whatever data you have about $f(x)$ or about the derivative $f'(x)$ and see what this implies. Usually, this quickly leads to whatever you are asked to show. $\square$

(4) **How do we find where $f(x)$ is increasing or decreasing?**

(a) Use the *Increasing/Decreasing Test.* If $f'(x) > 0$ everywhere on some interval, then $f(x)$ increases on this interval. If $f'(x) < 0$ everywhere on some interval, then $f(x)$ decreases on this interval. If $f'(x) = 0$ everywhere on some interval, then $f(x)$ is constant on this interval (by Cor.1.) Thus, find for which x's $f'(x) > 0$, for which $f'(x) < 0$.

(b) Be careful when $f'(x_0) = 0$: just the fact that $f'(x_0) = 0$ does **not** mean that the function is constant nearby x! (E.g. x^3 nearby $x = 0$.) Thus, check the sign of the derivative at nearby points and this will tell you what is going on exactly at $f'(x_0)$. For example, if $f'(x) > 0$ both on the left and right of x_0, then the function is increasing at x_0 too. But if $f'(x)$ changes sign at x_0, then $f(x_0)$ is a local extremum so that $f(x)$ is neither increasing nor decreasing at x_0. $\square$

(5) **How do we find the global extrema of $f(x)$?**

(a) First, find where $f'(x_0) = 0$ and record all such $f(x_0)$.

(b) Next, record all $f(x_0)$ where derivative does **not** exist, and also check and record what happens at the ends of the intervals (if there are any ends, that is.)

(c) Compare all values $f(x_0)$ at the critical points recorded above and find which are your global min/max. Be careful if the function is defined on an infinite interval or has vertical asymptotes. This gives opportunity for $f(x)$ to have arbitrarily large (small) values: some limit(s) of $f(x)$ could be $\pm\infty$, so in such cases, global min/max may not exist. $\square$

(6) **How do we find the local extrema of $f(x)$?** For simplicity of exposition, let's assume that $f(x)$ is a nice function, so we don't have to worry for now about discontinuities or non-existing derivatives.

(a) First find the potential local extrema of $f(x)$, i.e. check where $f'(x_0) = 0$ and record all such places x_0.

(b) Next, use the *Second Derivative Test* (if $f''(x_0)$ exists, of course) to determine which of these potential extrema are realized local extrema. In particular, if $f''(x_0) > 0$ then $f(x_0)$ is a local minimum; if $f''(x_0) < 0$ then $f(x_0)$ is a local maximum.

(c) If $f''(x_0) = 0$ or the second derivative $f''(x_0)$ does not exists (the latter could happen even if $f(x)$ is a nice function!), then the Second Derivative Test fails to give us anything useful, so we try the *First Derivative Test*. If $f'(x)$ changes its sign at x_0 from $+$ to $-$, then $f(x_0)$ is a local maximum; if $f'(x)$ changes its sign at x_0 from $-$ to $+$, then $f(x_0)$ is a local minimum. If $f'(x)$ does **not** change its sign at x_0, then $f(x_0)$ is **not** a local extremum. $\square$

(d) The *First Derivative Test* is **bulletproof!!!**

(7) **How do we find where the function is concave up or down?**

(a) Use the *Concavity Test*. If $f''(x) > 0$ everywhere on some interval, then $f(x)$ is concave-up on this interval. If $f''(x) < 0$ everywhere on some interval, then $f(x)$ is concave-down on this interval.

(b) If $f''(x_0) = 0$, then there could be an *inflection point* at x_0, so we check if $f''(x)$ changes sign at x_0. If yes, then at x_0 the function does indeed have an inflection point: the function changes from concave-up to concave-down or the other way around. But if $f''(x)$ does **not** change sign at x_0, then the function does **not** have an inflection point at x_0; and $f(x)$ continues to be concave-up (or concave-down) as it were before.

(c) As an example, compare the two functions x^3 and x^4 nearby $x_0 = 0$ and determine if they have inflection points at 0. Further, determine the intervals where these functions are concave-up or concave-down. $\square$

(8) **How do we use L'Hospital's Rule?** L'Hospital's Rule is applied to find limits **only** in cases where Limit Laws fail! Do **not** apply L'Hospital's Rule (LH) in cases where LLs work, or you will get an **incorrect** result. In other words, LH and LLs are *complements of each other*: one works exactly where the other doesn't work.

(a) The cases where LH applies can be determined as follows. Attempt to apply LLs and get an *indeterminate* of one of the types:

$$\frac{0}{0},\ \frac{\pm\infty}{\pm\infty},\ 0\cdot\infty,\ 0^0, \infty^0, 1^\infty.$$

None of these expressions make sense (and they cannot be given any reasonable uniform sense no matter how hard we try!)

(b) LH applies *directly* only to the first 2 types of indeterminants: $0/0$ and $\pm\infty/\pm\infty$. Thus, if we deal with a limit like:

$$\lim_{x\to a}\frac{f(x)}{g(x)} \overset{LL}{=} \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \overset{LL}{=} \frac{0}{0} \text{ or } \frac{\pm\infty}{\pm\infty},$$

we can replace each of $f(x)$ and $g(x)$ by their derivatives without changing the overall limit:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}. \quad \square$$

(c) Any of the other indeterminate forms $0\cdot\infty,\ 0^0, \infty^0, 1^\infty$ must be rewritten to fit one of the above two basic cases before we can even start applying LH.

- Product Indeterminacy $0\cdot\infty$ means that we have a limit like: $\lim_{x\to a} f(x)\cdot g(x) \overset{LL}{=} 0\cdot\infty$. We rewrite this simply by taking the reciprocal of $f(x)$ or $g(x)$, whichever is more convenient:

$$\lim_{x\to a} f(x)\cdot g(x) = \lim_{x\to a} \frac{g(x)}{\frac{1}{f(x)}} \stackrel{LL}{=} \frac{\pm\infty}{\pm\infty}, \text{ or } \lim_{x\to a} f(x)\cdot g(x) = \lim_{x\to a} \frac{f(x)}{\frac{1}{g(x)}} \stackrel{LL}{=} \frac{0}{0}.$$

Thus, we apply LH to the new rewritten expressions above. □

- Exponential Indeterminacies $0^0, \infty^0, 1^\infty$ mean that we have a limit like

$$\lim_{x\to a} f(x)^{g(x)} \stackrel{LL}{=} 0^0, \infty^0 \text{ or } 1^\infty.$$

This is a tougher case to handle. Recall the basic identity

$$e^{\ln\, blah} = blah, \text{ or equivalently } blah = e^{\ln blah}.$$

Setting $blah = f(x)^{g(x)}$, we rewrite:

$$f(x)^{g(x)} = e^{\ln\left(f(x)^{g(x)}\right)} = e^{g(x)\,\cdot\,\ln f(x)}.$$

Now instead of trying to find the limit of the whole expression, we can concentrate on finding the limit of the exponent only: $\lim_{x\to a} g(x)\cdot\ln f(x)$. If we apply LLs here, we will necessarily get an indeterminacy form $0\cdot\pm\infty$ or $\pm\infty\cdot 0$, with which we know how to deal from the previous discussion.

To summarize, to find $\lim_{x\to a} f(x)^{g(x)}$, where LLs would produce an indeterminacy of the form $0^0, \infty^0$ or 1^∞, we first rewrite the expression via an exponential as above. Then we find the limit $\lim_{x\to a} g(x)\cdot\ln f(x)$. Say, this limit turns out to be some number L. Therefore, the original limit will be e^L. □

(d) **Word of caution.** The idea of LH is to *simplify* our job of finding limits, **not** to complicate it! Thus, before you apply LH, find if LLs apply! Do not skip this step - this is not just an advice, this is a necessity! Only after you verify that LLs do **NOT** apply, you can start thinking of applying LH. And even then, it could be that LLs can get part of the job done - so use LLs as much as you can, and leave for LH to take care of the rest.

Why not just use LH on everything? Because LH may not apply; and even if it applies, if not used wisely it will yield a more complicated expression with which you will not know what to do! Remember: derivatives can be simpler than the original functions (e.g. polynomials), but sometimes derivatives can be much more complex than the original functions (e.g. often after you apply the Product Rule, Quotient Rule or Chain Rule).

So, LH is a powerful weapon, but only when used wisely! It can really backfire if we don't know how to use it!

(9) **How do we find slant asymptotes of $f(x)$?**

(a) Find the limits $\lim_{x\to\pm\infty} \frac{f(x)}{x}$. If it turns out that this limit does not exists, or exists but is not a finite number, then there will be no slant asymptote in the corresponding direction $x\to\infty$ or $x\to-\infty$. If this limit exists and is a finite number, then call it m:

$$m = \lim_{x\to\infty} \frac{f(x)}{x}.$$

m will be the slope of the slant asymptote.

(b) After we find the slope m, we proceed to find the constant b which will appear in the equation of the slant asymptote. To do that, find the limit

$$b = \lim_{x\to\infty} (f(x) - mx).$$

Thus, the slant asymptote in the direction $x\to\infty$ will be $y = mx + b$. □

Note that there is a short-cut for finding slant asymptotes in case we are looking at a fraction of two polynomials: $f(x)/g(x)$. For clarity, let $f(x) = Ax^m + \text{lower degree terms}$, and $g(x) = Bx^n + \text{lower degree terms}$; in other words, the leading coefficients of f and g are A and B, and their degrees are m and n respectively.

(a) If $m = n$ (i.e. $f(x)$ and $g(x)$ have the same degrees), then

$$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \frac{A}{B} \quad \text{(Why?)}$$

and hence $f(x)/g(x)$ will have a horizontal asymptote $y = A/B$ in *both* directions $x \to \pm\infty$.

(b) If $m < n$ (i.e. the degree of $f(x)$ is less than the degree of $g(x)$), then

$$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = 0 \quad \text{(Why?)}$$

and hence $f(x)/g(x)$ will have a horizontal asymptote $y = 0$ in *both* directions $x \to \pm\infty$.

(c) If $m = n+1$ (i.e. the degree of $f(x)$ is exactly 1 more than the degree of $g(x)$), then there will be a *slant* asymptote in both directions (why?) More precisely, to find this slant asymptote, we divide by x:

$$\lim_{x\to\pm\infty} \frac{f(x)}{xg(x)} = \frac{A}{B} \quad \text{(Why?)}$$

Hence the slant asymptote will have slope A/B in *both* directions $x \to \pm\infty$. To find the term b from linear equation of the slant asymptote, we proceed as follows:

$$b = \lim_{x\to\pm\infty} \frac{f(x)}{g(x)} - \frac{A}{B}x = \lim_{x\to\pm\infty} \frac{Bf(x) - Ax}{Bg(x)}.$$

There is a formula for this in terms of the second coefficients of $f(x)$ and $g(x)$, but it is not worth remembering. Instead, dutifully find the above limit and this will give your b in the slant asymptote formula.

(d) If $m > n+1$ (i.e. degree of $f(x)$ is 2 or more than the degree of $g(x)$), then there will be no horizontal or slant asymptotes (in any directions). The reason is that in this case:

$$\lim_{x\to\pm\infty} \frac{f(x)}{xg(x)} = \pm\infty$$

depending on whether the degrees of $f(x)$ and $g(x)$ are odd, even, etc. (Why?)

(e) In conclusion, if you are asked about horizontal and slant asymptotes for a fraction of two polynomials, you can use the above discussion. In particular, the only case where there will be a slant asymptote is when the degree of $f(x)$ is exactly 1 more than the degree of $g(x)$.

(10) **How do we sketch graphs of functions?** It will be very long to describe the whole process for an arbitrary function $f(x)$. On the exam, you will be given a specific function $f(x)$ to which some of the stuff below will apply, and some stuff will possibly be irrelevant.

(a) Establish where $f(x)$ is defined, where it is continuous and where it is differentiable.

(b) Establish if the function has any period. This is relevant usually in cases involving only trigonometric functions. If you find such a period, say, 2π, concentrate below only on one such interval, say $[0, 2\pi]$.

(c) Check if the function is odd or even - this will further reduce your work to concentrating only on, say, the interval $[0, +\infty)$. Such cases may occur with polynomials, fractions of polynomials, adn trig. functions. This step is not strictly necessary, but sometimes it is good to take it into account for more precise sketches.

(d) Find $f'(x)$ and $f''(x)$. Simplify them. If you get a fraction of polynomials, try to cancel everything in common and to factor top and bottom polynomials as much as possible.

(e) Establish where $f(x)$ is increasing and where it is decreasing, using $f'(x)$ and the relevant test(s).

(f) Find the local and global extrema (if any) of $f(x)$, using $f'(x)$, $f''(x)$ and the relevant test(s).

(g) Find where the function is concave-up and concave-down, and all inflection points, using $f''(x)$ and the relevant test(s).

(h) Find the vertical asymptotes (if any). They will appear where $f(x)$ has an infinite (at least) one-sided limit, i.e. if $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$; then the vertical line $x = a$ is such an asymptote. If $f(x)$ is a fraction, such solutions will be produced by the roots of the denominator.

(i) Find the horizontal asymptotes: these will appear where $f(x)$ has a finite limit when $x \to \pm\infty$, i.e. if $\lim_{x\to\pm\infty} f(x) = L$; then the horizontal line $y = L$ is such an asymptote. If $f(x)$ is a fraction, both numerator and denominator will be involved in this step. Note that L'Hospital's Rule may or may not be needed here.

(j) Find the slant asymptotes (if any). Note that L'Hospital's Rule may or may not be needed here.

(k) If possible, find the roots of the function, i.e. its x-intercepts: try to solve $f(x) = 0$ if possible. If $f(x)$ is a fraction, such solutions will be produced in the numerator. (The denominator will be irrelevant in this step.) It is also good to find the y-intercept, by setting $x = 0$ in $f(x)$.

(l) The above information should be enough to sketch a very good graph of $f(x)$.

- On the x-axis, mark all interesting points which you found above, mark the intervals where $f(x)$ is increasing or decreasing (or constant, for that matter). Mark the x-intercepts and the y-intercept (if any). Mark any local and global extrema you found. Mark any inflection points you found.
- Draw the vertical, horizontal and slant asymptotes (if any). Make a mental picture of what is happening nearby each of these asymptotes. Be careful nearby the vertical asymptotes to reflect whether a given one-sided limit is $+\infty$ or $-\infty$.
- At this moment, if you are not sure how the graph looks in some intervals, plot a few other points there.
- All that is left is to connect the points you plotted respecting the properties you found above: domain of definition, increase/decrease, extrema, concavity and inflection points, dis-continuity and non-differentiability, asymptotes, period of function, odd/even function. $\square$

(11) **How do we cook up a function $g(x)$ which captures the sign of $f'(x)$ or of $f''(x)$?**

Speaking of shortcuts, this may be the single most significant shortcut in drawing graphs: when applicable, this shortcut may allow you to establish all intervals of increase, decrease, concavity, loc. min's and max's **without** having to plug in a single value into $f'(x)$ and $f''(x)$! Now, that's what **I** call a shortcut!

An example illustrates the technique. Say, $f'(x)$ equals the following stupendous expression:

$$f'(x) = \frac{4(x-1)^3(x+7)^6(x-4)^{13}(x^2+10)}{(x+2)^5(x-2)^2(2x^2+3)^2(x+8)}.$$

Naturally, we are interested where this expression is positive, negative and zero. Now, you can plug in values into $f'(x)$ and fill in a table with $+$ and $-$ signs until you are blue in the face, but ... let's do it the **cool way**!

(a) First of all, find where $f'(x) = 0$. This comes from the numerator: $x = 1, -7, 4$. These are the **potential** local min's and max's of the original function $f(x)$. Note that the factor $x^2 + 10$ does not contribute any zeros (why?)

(b) Let's not forget where $f'(x)$ is **not** defined, just so that we don't write meaningless things later on: this comes from the denominator, i.e. $x \neq -2, 2, -8$. Note that $2x^2 + 3$ does not mess the domain of $f'(x)$ (why?)

(c) Now, we throw out all **irrelevant** terms in $f'(x)$, about which we already know that they are **always** positive. In our case, $(x+7)^6$, $x^2 + 10$, $(x-2)^2$, $(2x^2+3)^2$ and 4 can all go, since they are always positive. We can't quite throw out $(x-1)^3$ because it can be either positive or negative, but we can throw out a big chunk of it: $(x-1)^3 = (x-1)^2 \cdot (x-1)$, so $(x-1)^2$ can happily go, leaving only $(x-1)$. In a similar fashion, let's get rid of $(x-4)^{12}$, leaving $(x-4)$ in the numerator.
In the denominator, we have $(x+2)^5$. Instead of throwing out $(x+2)^4$ and leaving $(x+2)$ in the denominator, it is better to multiply top and bottom by an extra $(x+2)$:

$$\frac{1}{(x+2)^5} = \frac{(x+2)}{(x+2)^6}.$$

Now, we happily throw out $(x+2)^6$, and we are left with $(x+2)$ in the **numerator**! In a similar vein, multiply top and bottom by $(x+8)$, throw out $(x+8)^2$ from the denominator, leaving $(x+8)$ in the numerator.
We have created a monster! No, just kidding. We have created a new simple function $g(x)$, which completely captures the sign of $f'(x)$:

$$g(x) = (x-1)(x-4)(x+2)(x+8).$$

To see where $g(x)$ is positive and where it is negative, simply arrange all zeros $x = 1, 4, -2, -8$ of $g(x)$ on the number line and note that $g(x)$ is a **polynomial** of degree 4 with **positive** leading coefficient (in our case, the leading term is x^4). Then, start from the very right top corner of the graph paper, go down to the first zero of $g(x)$ ($x = 4$ in our case), and then start alternating going above and below the x–axis between the zeros:

We conclude that $f'(x) > 0$ when $x \in (-\infty, -8) \cup (-2, 1) \cup (4, \infty)$, and $f'(x) < 0$ when $x \in (-8, -2) \cup (1, 4)$.

(d) How about the potential local min's and max's? These were at $x = 1, -7, 4$. A **common mistake** is to include here $x = -8, -2$, but ... these are zeros of $g(x)$, and we **don't care** where $g(x) = 0$! $g(x)$ exists with one and only one purpose: to tell us where $f'(x)$ is positive or negative, but **not** where $f'(x) = 0$! So, be very careful here!
OK, at $x = 1$ f' changes sign from $+$ to $-$, hence $f(x)$ has a local maximum at $x = 1$; at $x = 4$ f' changes sign from $-$ to $+$, hence $f(x)$ has a local minimum at $x = 4$. However, at $x = 7$ $f'(x)$ doesn't change sign: it stays negative, so no local extrema at $x = 7$ - the function $f(x)$ simply levels off at $x = 7$ but continues to decrease there.

(e) Let's summarize. $f(x)$ has several interesting points: $x = 1$ (local maximum), $x = 4$ (local minimum), $x = -8, -2, 2$ (vertical asymptotes). Even without having studied yet $f''(x)$, nor having yet found any slant or horizontal asymptotes of $f(x)$, we can draw an approximate graph of $f(x)$:

To check that $f(x)$ indeed has the above concavities and find the inflection points, one needs to look at $f''(x)$. Note that one needs to evaluate the actual $f(1)$ and $f(4)$ - I have drawn some hypothetical points here, since I don't have the original function $f(x)$.

(12) **Summary of Derivative Tests**

#	Test	Uses	Tells us about $f(x)$	Remarks
1	Monotone Test	$f'(x) > 0$ or $f'(x) < 0$	$f(x)$ is increasing or decreasing	bullet–proof
2	1st Derivative Test	$f'(x_0) = 0$, and $f'(x)$ changes sign at x_0	potential local min/max at x_0, realized local min/max at x_0	bullet–proof
3	2nd Derivative Test	$f'(x_0) = 0$, and $f''(x_0) > 0$ or $f''(x_0) < 0$	potential local min/max at x_0, realized local min/max at x_0	fails if $f'(x_0) = 0 = f''(x_0)$
4	Concavity Test	$f''(x) > 0$ or $f''(x) < 0$	$f(x)$ is concave up or down	bullet–proof
5	Inflection Pt Test	$f''(x_0) = 0$, and $f''(x)$ changes sign at x_0	$f(x)$ has inflection point at x_0	$f''(x_0) = 0$ alone does not imply inflection pt at x_0

(13) **How do we solve optimization problems?** Optimization problems come in great varieties of themes, and each problem requires individual consideration. Therefore, there is no uniform way of solving these problems. Below we describe the general *strategy* for solving optimization problems.

(a) Let the optimization problem we are considering be named Problem 1.

(b) Translate Problem 1 into a mathematical Problem 2. For this, one has to understand very well the original Problem 1, and use whatever means necessary for the mathematical translation. Usually, one ends up with a function $f(x)$, and is being asked to find its global or local extrema.

(c) Solve the mathematical Problem 2 using the techniques learned in Calculus I.

(d) Translate the math answer in Problem 2 into a practical answer in the original Problem 1. $\square$

(14) **What is the "Distance Formula" and how do we use it?**

(a) The *Distance Formula* (DF) tells us the distance between two known points, i.e. we know their coordinates. Thus, if $A(x_1, y_1)$ and $B(x_2, y_2)$, the distance between A and B will be

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

(b) The *Distance Formula* often comes in handy in Optimization Problems. Say, for example, we are on a deck of a ship. The front portion of the deck is in the shape of a parabola $y = x^2$. Suppose a person is drowning in the water at location $(18, 0)$. Where should we stand on the deck of the ship and throw a life-saving device?

Obviously, we should stand at the place on the deck which is closest to the drowning person. OK, we don't know where this place will be, but for sure it must be on the edge of the deck! Thus, if we stand somewhere on the edge of the deck, we are standing at some point (x, x^2) -

because of the parabolic shape of the deck. The distance between us and the drowning person is given by:

$$d(x) = \sqrt{(x-18)^2 + (x^2-0)^2} = \sqrt{(x-18)^2 + x^4}.$$

Therefore, all we have to do is find the minimum of this function and where this minimum occurs!

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However, an important simplification is to notice that $d(x)$ will be minimal exactly when its square $d^2(x)$ is minimal. This allows us to get rid of the uncomfortable square in this case and deal with much simpler derivatives. Thus, we set out to find the global minimum of $f(x) = (x-18)^2 + x^4$.

$$f'(x) = 2(x-18) + 4x^3 = 2(2x^3 + x - 18).$$

We are interested where $f'(x) = 0$, i.e. $2x^3 + x - 18 = 0$. Let's hope that we won't have to solve such a complicated equation on the test, but for now: notice that $x = 2$ is a root if this equation, so that $(x-2)$ must factor out. Indeed: $2x^3 + x - 18 = (x-2)(2x^2+4x+9)$ (check it out!) The quadratic factor $2x^2+4x+9$ has no zeros, as the quadratic formula yields a negative number under a square root (check this too!). Thus, the only critical point of $f(x)$ is at $x = 2$.

To determine if $f(2)$ is indeed a local extremum, we could find $f''(x)$: $f''(x) = 6x^2 + 1 > 0$ always. In particular, $f(2) > 0$, and indeed $f(x)$ has a local minimum at $x = 2$. Moreover, since $2x^2 + 4x + 9$ is always positive – it is a concave up parabola with no zeros!! – the sign of $f'(x) = (x-2)(2x^2+4x+9)$ is captured by $x - 2$. Hence, $f'(x) > 0$ for $x > 2$, and $f'(x) < 2$ for $x < 2$, which again confirms that $f(x)$ has a local (and global!) minimum at $x = 2$.

Correspondingly, the distance $d(x)$ is minimum when $x = 2$. This is what we were after: we should position ourselves at point $(2, 4)$ on the deck so that we are closest to the drowning person. The actual distance will be $\sqrt{(18-2)^2 + (4-0)^2} = \sqrt{256+16} = \sqrt{272} \sim 16.5$m. $\square$

(15) **How do we use area formulas in optimization problems?**

(a) First, let's see what area formulas we know. The most basic ones are:

- Area of a square with side x is x^2.
- Area of a rectangle with sides x and y is $x \cdot y$.
- Area of a triangle with base x and height h is $\dfrac{x \cdot h}{2}$.
- Area of a circle with radius r is πr^2.

(b) It could happen that we also need formulas for the *perimeter* of the above basic figures. (If we take a walk along the sides/edges of a figure, the total length of the walk is called *the perimeter* of the figure.)

- Perimeter of a square with side x is $4x$.

- Perimeter of a rectangle with sides x and y is $2x + 2y$.
- Perimeter of a triangle with sides x, y and z is $x + y + z$.
- Perimeter of a circle with radius r is $2\pi r$.

(c) See if your optimization problem is asking for a maximum or minimum area. (The word "area" may not be mentioned, but the problem could still boil down to an "area" of some sort.) Find the applicable area formula(s), and create a function $f(x)$ describing the given problem.

(d) For example, say we want to make flower and vegetable gardens up in the Berkeley Hills, and put up a fence around them to prevent deer from eating by mistake all our carnations and juicy Bulgarian tomatoes. We want a circular flower garden, and a square vegetable garden. Unfortunately, we aren't rich, yet the fence costs about \$50 per yard! After long soul (and pocket) searching we decide that we can afford at most \$1000 for the fence. What sizes should we choose for the flower and vegetable gardens so that the total garden area is as large as possible?

- The whole story about the price of the fence simply means that we'll have at most a 20 yard fence total for both gardens. If we want to encompass the largest amount of ground, then we must use the longest possible fence - this much is obvious without any higher mathematics.

- The problem is talking about shapes and areas of gardens, hence let's see what area formulas will be relevant: the area of a square x^2 and the area of a circle πr^2. However, the problem is also talking about fences, hence the useful perimeter formulas will be $4x$ for a square and $2\pi r$ for a circle.
- Let's assign letters to the hypothetical sizes of the two gardens: say, the square garden will have size x yards and the circular garden will have radius r yards. The total area of the two gardens will be: $x^2 + \pi r^2$. We want to maximize this quantity.
- The fence is, as we decided above, 20 yards total. The total perimeter of the two gardens will be $4x + 2\pi r$. Hence we want $20 = 4x + 2\pi r$. This means that the size x and the radius r are not completely independent of each other! For example, we can solve for x:
$$x = \frac{20 - 2\pi r}{4} = 5 - \frac{\pi r}{2}.$$
- So, let's substitute x is the formula for the total area above: $(5 - \frac{\pi r}{2})^2 + \pi r^2$. Simplifying:
$$f(r) = \left(\frac{\pi^2}{4} + \pi\right) r^2 - 5\pi r + 25.$$
Note that this is a function in *one variable only*, namely r! And not a complicated function at all - just a quadratic polynomial. We want to maximize it.

- $f'(r) = \dfrac{\pi^2 + 4\pi}{2} r - 5\pi$. Thus, $f'(r) = 0$ if $r = \frac{10}{\pi+4}$ (Why? Do it yourself.) To make sure that this is indeed a local extremum, we check $f''(r) = \frac{\pi^2+4\pi}{2}$, which is always positive. Hence this is a local (and global in this case) minimum. But we wanted the global **maximum**! Our parabola decreases until it hits the above local minimum, and then increases. Hence its global maximum will be attained at the ends of the interval.

- What interval are we talking about? A parabola is defined everywhere! However, the real situation poses some restrictions. For example, we can't have negative x or r - they must both be ≥ 0. Since $x = 5 - \frac{\pi r}{2} \geq 0$, we discover that $r \leq 10/\pi$, thus the interval of r is $[0, 10/\pi]$.
- Thus, $f(x)$ will attain its maximum at one of the ends of $[0, 10/\pi]$: $f(0) = 25$, and $f(10/\pi) = 100/\pi \sim 31.8$. The second case obviously wins, but what does it mean? To have maximal total area we must have $r = 10/\pi$ and $x = 5 - \frac{\pi r}{2} = 5 - \frac{\pi 10}{2\pi} = 0$. In real life, our answer means that if we really want to maximize the used area, we will have no square garden, and the whole land (and fence) should go into a circular flower garden of radius $10/\pi \sim 3.18$ yards.
- A further discussion is due here. We see that if we want to have some vegetable garden too, we have to change somehow our requirements in the original problem! We simply cannot have BOTH 20 yards of fence AND maximal total area, and still expect that we will have a square vegetable garden! Under these conditions, the circular garden will win no matter what, and it will use up all the ground! $\square$
- Suppose instead that we had already bought the 20 yard fence, and wanted to use all of it (well, we won't throw it away!), but we had a shortage of space, so we wanted to use as little space for both gardens as possible. The set-up of the function $f(r)$ will be exactly the same as above, except that we will be looking for its minimum.
  We already found where this happens: at $r = \frac{10}{\pi+4} \sim 1.4$, and correspondingly, $x = 5 - \frac{\pi r}{2} = \frac{20}{\pi+4} \sim 2.8$. These will be two jolly small gardens: a circular garden of radius about 1.4 yards, and a square garden of size about 2.8 yards... But we get what we asked for: the total fence used will be exactly 20 yards, and the total area used will be as small as possible. $\square$

4. Useful Formulas and Miscellaneous Facts

(1) Pythagorus Theorem for a right triangle.
(2) Quadratic formula.
(3) Fraction manipulations, exponential and logarithmic manipulations and formulas.
(4) LLs, DLs.

5. Cheat Sheet and Studying for the Exam

For the exam, you are allowed to have a "cheat sheet" - *one page* of a regular 8×11 sheet. You can write whatever you wish there, under the following conditions:

- The whole cheat sheet must be **handwritten by your own hand**! No xeroxing, no copying, (and for that matter, no tearing pages from the textbook and pasting them onto your cheat sheet.)
- Any violation of these rules will disqualify your cheat sheet and may end in your own disqualification from the midterm. I may decide to randomly check your cheat sheets, so let's play it fair and square. :)
- Don't be a **freakasaurus**! Start studying for the exam several days in advance, and prepare your cheat sheet at least 2 days in advance. This will give you enough time to become familiar with your cheat sheet and be able to use it more efficiently on the exam.
- **Do NOT overstudy on the day of the exam!! More than 3 hours of math study on the day of the Final is counterproductive! No kidding!**