Topics for Review for Midterm I in Calculus 1A

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1. Definitions

Be able to **write** precise definitions for any of the following concepts (where appropriate: both in words and in symbols), to **give** examples of each definition, to **prove** that these definitions are satisfied in specific examples. Wherever appropriate, be able to **graph** examples for each definition. What is/are:

(1) a *function*?
(2) an *independent* variable; a *dependent* variable?
(3) a *domain*; a *range* of a function?
(4) the *graph* of a function? the vertical line test?
(5) a *piecewise defined* function?
(6) the function *absolute value* $|x|$?
(7) an *increasing* function; a *decreasing* function?
(8) an *odd* function; an *even* function?
(9) a *linear*, *quadratic*, *polynomial*, *rational* function? domains of these functions?
(10) a *power*, *exponential*, *logarithmic* function? domains of these functions?
(11) a *trigonometric* function: $\sin x$, $\cos x$, $\tan x$, $\cot x$? domains of these functions?

Example. The function $\cot x$ is defined as follows: take a right triangle ABC with $\angle A = x$ (in radians) and $\angle B = \pi/2$; then set $\cot x = AB/BC$ (adjacent leg/opposite leg). Alternatively, let an angle of measure x be centered at the origin O and its first arm coincide with the positive direction of the x-axis. Then $\cot x$ is the horizontal displacement on the line $y = 1$ made by the second arm of the angle. (Check your session notes to review this second definition!)

(12) the *tangent line* to the graph of a function $f(x)$ at $x = a$? a *secant line* of the graph of $f(x)$? what does it mean that $f(x)$ has a *vertical* tangent at $x = a$?
(13) the *average velocity* and the *instantaneous velocity* of an object whose movement is given by $f(t)$?
(14) the *derivative* $f'(a)$ at $x = a$? What does it mean that $f(x)$ is *differentiable* at a?
(15) the *derivative function* $f'(x)$? What does it mean that $f(x)$ is *differentiable* on (A, B)?
(16) the notation $\dfrac{d}{dx}$?
(17) the number e? the properties of e? e^x? What is $\ln x$?
(18) the *composition* $f \circ g$ of two functions $f(x)$ and $g(x)$?
(19) the *inverse* function $f^{-1}(x)$ of a function $f(x)$? How can we obtain their graphs from each other? What is the relationship between their derivatives?
(20) *implicit* differentiation?
(21) *logarithmic* differentiation? *exponential* differentiation?

(22) *Limit* of $\lim_{x\to\Box_1} f(x) = \Box_2$ where each of the "boxes" can be a finite number or $\pm\infty$?

Example. To say that $\lim_{x\to-\infty} f(x) = 10$ means that:

(a) $f(x)$ can be made as close to 10 as we please, provided x is small enough.
(b) every ϵ-goal around 10 can be achieved by $f(x)$ provided x is small enough.
(c) $\forall\epsilon > 0$, $f(x)$ falls within ϵ of 10 provided x is small enough.
(d) $\forall\epsilon > 0\ \exists M < 0$ such that $|f(x) - 10| < \epsilon$ whenever $x < M$.

(23) *One-sided limit*, e.g. $\lim_{x\to a^-} f(x) = L$, $\lim_{x\to a^+} f(x) = -\infty$, etc.?

Example. To say that $\lim_{x\to 4^+} f(x) = 7$ means that

(a) $f(x)$ can be made as close to 7 as we please, provided x is close enough to 4 and $x > 4$.
(b) every ϵ-goal around 7 can be achieved by $f(x)$ provided x is close enough to 4 and $x > 4$.
(c) $\forall\epsilon > 0$, $f(x)$ falls within ϵ of 7 provided x is close enough to 4 and $x > 4$.
(d) $\forall\epsilon > 0\ \exists\delta > 0$ such that $|f(x) - 7| < \epsilon$ whenever $4 < x < 4 + \delta$.

(24) *Vertical asymptote* of $f(x)$ at a? *Horizontal asymptote* of $f(x)$?

(25) *Continuity. What does it mean that a function $f(x)$:*

(a) is *continuous at a*?
(b) is *continuous from the right (or from the left) at a*?
(c) is *continuous on* (a, b), $[a, b)$, $[a, b]$, $(-\infty, b]$, (a, ∞), etc.
(d) has a *removable discontinuity* at a? *jump discontinuity* at a? *infinite discontinuity* at a?

2. Theorems

Be able to **write** what each of the following theorems (laws, propositions, corollaries, etc.) says. Be sure to understand, distinguish and **state** the conditions (hypothesis) of each theorem and its conclusion. Be prepared to **give** examples for each theorem, and most importantly, to **apply** each theorem appropriately in problems. The latter means: decide which theorem to use, check (in writing!) that all conditions of your theorem are satisfied in the problem in question, and then state (in writing!) the conclusion of the theorem using the specifics of your problem.

(1) The *Squeeze (Sandwich, Policemen)* Theorem.

(2) The *Intermediate Value* Theorem (IVT)

Example. The IVT says the following.[1] Let $f(x)$ be a function defined and continuous on an interval $[A, B]$. Let N be an intermediate value for $f(x)$, i.e. N is between $f(A)$ and $f(B)$:

$$f(A) \le N \le f(B) \text{ or } f(A) \ge N \ge f(B).$$

Then somewhere on (A, B) the function $f(x)$ attains the value N, i.e. there exists some $x_0 \in (A, B)$ such that $f(x_0) = N$.

(3) *(Finite) Limits Laws (LLs)* (Section 2.3): addition, subtraction, multiplication, division, basic examples, multiplication by a constant, powers, roots. (Be careful about the division law! What extra conditions does it require?) $x \to \Box$ means: $x \to a$, $x \to \infty$, or $x \to -\infty$.

[1] In order to apply IVT, one must first check (and write) the two conditions: that $f(x)$ is continuous on $[A, B]$, and that N is indeed an intermediate value between $f(A)$ and $f(B)$; only after that, one states the conclusion of the IVT for the particular $f(x)$ in question.

#	Theorem	Hypothesis	Conclusion
1	Linearity	$\lim\limits_{x\to\square} f_1(x) = L_1,\ \lim\limits_{x\to\square} f_2(x) = L_2,\ c_1, c_2 \in \mathbb{R}$	$\lim\limits_{x\to\square}(c_1 f_1(x)+c_2 f_2(x))=c_1 L_1+c_2 L_2$
2	Product	$\lim\limits_{x\to\square} f_1(x) = L_1,\ \lim\limits_{x\to\square} f_2(x) = L_2$	$\lim\limits_{x\to\square} f_1(x) f_2(x) = L_1 L_2$
3	Quotient	$\lim\limits_{x\to\square} f_1(x) = L_1,\ \lim\limits_{x\to\square} f_2(x) = L_2,\ f_2(x) \neq 0$ for $x \approx a,\ L_2 \neq 0$	$\lim\limits_{x\to\square} \frac{f_1(x)}{f_2(x)} = \frac{L_1}{L_2}$
4	∞-Limit Sum	$\lim\limits_{x\to\square} f_1(x) = \infty$, and $\lim\limits_{x\to\square} f_2(x) = \infty$ or $f_2(x)$ bounded below	$\lim\limits_{x\to\square}(f_1(x) + f_2(x)) = \infty$
5	∞-Limit Product	$\lim\limits_{x\to\square} f_1(x) = \infty$, and $\lim\limits_{x\to\square} f_2(x) = L > 0$ or $f_2(x)$ bounded below by $K > 0$	$\lim\limits_{x\to\square} f_1(x) f_2(x) = \infty$
6	∞-Reciprocal	$\lim\limits_{x\to\square} f(x) = \infty$	$\lim\limits_{x\to\square} \frac{1}{f(x)} = 0$
7	0-Reciprocal	$\lim\limits_{x\to\square} \frac{1}{f(x)} = 0$ and $f(x) > 0$	$\lim\limits_{x\to\square} f(x) = \infty$
8	Sandwich, Policemen	$f(x) \leq g(x) \leq h(x)$ for $x \approx \square$ and $\lim\limits_{x\to\square} f(x) = L = \lim\limits_{x\to\square} h(x)$	$\lim\limits_{x\to\square} g(x) = L$
9	∞-Sandwich	$f(x) \leq g(x)$ for $x \approx \square$ and $\lim\limits_{x\to\square} f(x) = \infty$	$\lim\limits_{x\to\square} g(x) = \infty$
10	Lim-Location	$f(x) \leq M$ for $x \approx \square$ and $\lim\limits_{x\to\square} f(x)$ exists,	$\lim\limits_{x\to\square} f(x) \leq M$
11	Lim-Location	$f(x) \leq g(x)$ for $x \approx \square$, $\exists \lim\limits_{x\to\square} f(x)$, $\exists \lim\limits_{x\to\square} g(x)$	$\lim\limits_{x\to\square} f(x) \leq \lim\limits_{x\to\square} g(x)$
12	Fun-Location	$\lim\limits_{x\to\square} f(x) = L < M$	$f(x) < M$ for $x \approx \square$

(4) *Infinite Limit Laws (∞–LLs).*

Name	∞-LL: Formula	Example
1. addition	$\infty + \infty = \infty$	$\lim\limits_{x\to\infty}(x + x^2) = \lim\limits_{x\to\infty} x + \lim\limits_{x\to\infty} x^2 = \infty + \infty \overset{\infty\text{LL}}{=} \infty$
	$(-\infty) + (-\infty) = -\infty$	$\lim\limits_{x\to\infty}(-x - x^2) = \lim\limits_{x\to\infty}(-x) + \lim\limits_{x\to\infty}(-x^2) = (-\infty) + (-\infty) \overset{\infty\text{LL}}{=} -\infty$
	$\infty - \infty$ undefined	**Never use ∞-LLs in such cases.**
2. multiplication	$\infty \cdot \infty = \infty$	$\lim\limits_{x\to\infty} x^2 = \lim\limits_{x\to\infty} x \cdot \lim\limits_{x\to\infty} x = \infty \cdot \infty \overset{\infty\text{LL}}{=} \infty$
	$(-\infty) \cdot (-\infty) = \infty$	$\lim\limits_{x\to\infty}(-x)^2 = \lim\limits_{x\to\infty}(-x) \cdot \lim\limits_{x\to\infty}(-x) = (-\infty) \cdot (-\infty) \overset{\infty\text{LL}}{=} \infty$
	$\infty \cdot (-\infty) = -\infty$	$\lim\limits_{x\to\infty} -x^2 = \lim\limits_{x\to\infty} x \cdot \lim\limits_{x\to\infty}(-x) = \infty \cdot (-\infty) \overset{\infty\text{LL}}{=} -\infty$
3. multiplication by constant	$c \cdot \infty = +\infty$ if $c > 0$	$\lim\limits_{x\to\infty}(2x) = 2 \lim\limits_{x\to\infty} x = 2 \cdot \infty \overset{\infty\text{LL}}{=} \infty$
	$c \cdot \infty = -\infty$ if $c < 0$	$\lim\limits_{x\to\infty}(-2x) = -2 \lim\limits_{x\to\infty} x = -2 \cdot \infty \overset{\infty\text{LL}}{=} -\infty$
	$0 \cdot \infty$ undefined	**Never use ∞-LLs in such cases.**
4. addition by constant	$c + \infty = +\infty\ \forall c$	$\lim\limits_{x\to\infty}(-2 + x) = -2 + \lim\limits_{x\to\infty} x = -2 + \infty \overset{\infty\text{LL}}{=} \infty$
	$c - \infty = -\infty\ \forall c$	$\lim\limits_{x\to\infty}(-2 - x) = -2 - \lim\limits_{x\to\infty} x = -2 - \infty \overset{\infty\text{LL}}{=} -\infty$
5. division by $\pm\infty$	$\frac{c}{+\infty} = 0\ \forall c$	$\lim\limits_{x\to\infty} \frac{2}{x} = \frac{2}{\lim\limits_{x\to\infty} x} = \frac{2}{\infty} \overset{\infty\text{LL}}{=} 0$
	$\frac{c}{-\infty} = 0\ \forall c$	$\lim\limits_{x\to\infty} \frac{2}{-x} = \frac{2}{\lim\limits_{x\to\infty}(-x)} = \frac{2}{\infty} \overset{\infty\text{LL}}{=} 0$
6. division by $0^{\pm}$	$\frac{c}{0^+} = +\infty\ \forall c > 0$	$\lim\limits_{x\to 0} \frac{2}{x^2} = \frac{2}{0^+} \overset{\infty\text{LL}}{=} +\infty$, $\lim\limits_{x\to 0} \frac{-2}{x^2} = \frac{-2}{0^+} \overset{\infty\text{LL}}{=} -\infty$
	$\frac{c}{0^-} = -\infty\ \forall c > 0$	$\lim\limits_{x\to 0} \frac{2}{-x^2} = \frac{2}{0^-} \overset{\infty\text{LL}}{=} -\infty$, $\lim\limits_{x\to 0} \frac{-2}{-x^2} = \frac{-2}{0^-} \overset{\infty\text{LL}}{=} +\infty$
	$\frac{0}{0}, \frac{c}{0}$ undefined $\forall c$	**Never use ∞-LLs in such cases.**
7. basic cases	$\lim\limits_{x\to\infty} \frac{1}{x} = 0,\ \lim\limits_{x\to-\infty} \frac{1}{x} = 0$	
	$\lim\limits_{x\to\infty} x = \infty,\ \lim\limits_{x\to-\infty} x = -\infty$	
	$\lim\limits_{x\to 0^+} \frac{1}{x} = +\infty,\ \lim\limits_{x\to 0^-} \frac{1}{x} = -\infty,\ \lim\limits_{x\to 0} \frac{1}{x}$ **does not exist.**	

In the infinite limit laws, an expression like "$(-\infty)+(-\infty)=-\infty$" does **not** have a meaning on its own, except *in context*, i.e. it refers only to the following situation and to nothing else:

Theorem "$(-\infty)+(-\infty)=-\infty$". "If for functions $f(x)$ and $g(x)$ we know that $\lim_{x\to\Box} f(x)=-\infty$, $\lim_{x\to\Box} g(x)=-\infty$, then $f(x)+g(x)$ also has a limit when $x\to\Box$: this limit is $\lim_{x\to\Box}(f(x)+g(x))=-\infty$."

Note that there are no infinite limit laws for substraction, division, or multilication of the type $0\cdot\infty$, i.e. the symbolic expressions $\infty-\infty, \infty/\infty, 0\cdot\infty$ do not make sense, and they are called *indeterminate*.

(5) *Continuity Laws (CLs).* Hypothesis for all continuity theorems below: If $f(x)$ and $g(x)$ are *continuous* at $x=a$, i.e. $\lim_{x\to a} f(x)=f(a)$ and $\lim_{x\to a} g(x)=g(a)$, then

#	Theorem Name	Conclusion	Follows from
1	Continuity of Sum	$f(x)+g(x)$ is also continuous at $x=a$	LL for sum
2	Continuity of Difference	$f(x)-g(x)$ is also continuous at $x=a$	LL for difference
3	Continuity of Product	$f(x)g(x)$ is also continuous at $x=a$	LL for product
4	Continuity of Ratio ($g(a)\neq 0$)	$f(x)/g(x)$ is also continuous at $x=a$	LL for ratio
5	Continuity Respects Constants	$c\cdot f(x)$ is also continuous at $x=a$	LL for jumping constants
6	Continuity of Composition: $h(x)$ is continuous at $b=f(a)$	$h(f(x))$ is also continuous at $x=a$	LL for composition
7	Continuity & Positivity: $f(a)>0$	$f(x)>0$ for $x\approx a$	Limit Definition for $f(x)$

Note that all *Continuity Laws (CLs)* follow from the corresponding *Limit Laws (LLs)*. The CLs above allow us to perform algebraic operations (and compositions) on continuous functions. Thus, we can construct more complex continuous functions from simpler continuous functions. To do this, we need to have a starting collection of

(6) *Basic Continuous Functions.* All of the following types of functions are *continuous* on their respective domains of definition:

#	Function	Algebraic Formula and Conclusion	Follows from
1	Constants	c continuous at $\forall x$	LL for constants
2	Linear	$ax+b$ continuous at $\forall x$	LL for linear fn's
3	Quadratic	ax^2+bx+c continuous at $\forall x$	LL for quadratic fn's
4	Power	x^n continuous at $\forall x$, $\forall n=1,2,3,\ldots$	LL for powers
5	Polynomial	$a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0$ continuous at $\forall x$	LL for polynomials
6	Rational	$\dfrac{f(x)}{g(x)}$ continuous where $g(x)\neq 0$ ($f(x),g(x)$ - poly's)	LL for ratio, CL for poly's
7	Root	$\sqrt[n]{x}$ continuous at $\forall x$ where defined	LL for roots
8	Exponential	a^x continuous at $\forall x$ $(a>0)$	LL for exponentials
9	Logarithmic	$\log_a x,\ \ln x$ continuous at $\forall x>0$ $(a>0)$	LL for logarithmics
10	Trigonometric	$\sin x,\ \cos x$ continuous at $\forall x$; $\tan x,\ \cot x$ cont. on domain: $\tan x$: $x\neq\pm\pi/2,\pm3\pi/2,\pm5\pi/2\ldots,(2n+1)\pi/2$, $\cot x$: $x\neq 0,\pm\pi,\pm2\pi,\pm3\pi\ldots,n\pi,\ n\in\mathbb{Z}$	LL for trig. fn's
11	Inverse Trigonometric	arcsin$x$, arccos$x$, arcot$x$, arccot$x$ continuous on domain: arcsin$x$: $[-1,1]\to[-\pi/2,\pi/2]$, arccos$x$: $[-1,1]\to[0,\pi]$, arcot$x:\mathbb{R}\to(-\pi/2,\pi/2)$, arccot$x:\mathbb{R}\to(0,\pi)$	LL for inverse trig. fn's

(7) *Special Continuity Theorems*: continuity statements about various types of functions, e.g. polynomials, rational, root, trigonometric, exponential and logarithmic functions. (Be careful to include the domain of definition of these functions.)

Example. The *logarithmic* function $\log_c x$ is defined for any $x>0$ (the base of the logarithm, c, must also be positive.) Any logarithmic function is continuous everywhere on its domain of definition, i.e. on $(0,+\infty)$.

(8) *"Double-sided" Theorems for Limits and Continuity.*
Example. The limit of $f(x)$ when $x \to a$ exists if and only if the two one-sided limits of $f(x)$ at a exist and are equal to each other. In this case, the limit of $f(x)$ exists and equals their common value. In symbols: $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$.

(9) *Theorem I.(Differentiable $\Rightarrow$ continuous.)* If $f(x)$ is differentiable at a, then $f(x)$ is continuous at a. If $f(x)$ is differentiable everywhere on its domain, then it is also continuous everywhere on its domain.

(10) *Contrapositive Theorem II. (Non-differentiable $\Rightarrow$ non-continuous.)* If $f(x)$ is *not* continuous at a, then $f(x)$ is *not* differentiable at a.

(11) *Converse Statement is False!* Continuity does not guarantee differentiability. Counterexample?

(12) *Differentiation Laws (DLs).*

(a) *Basis Cases.*

(i) *Constant Functions:* $(c)' = 0$ for any constant c.

(ii) *Power Rule:* $(x^c)' = c\,x^{c-1}$ for any constant c.

(iii) *Natural Exponential & Logarithmic Functions:* $(e^x)' = e^x$; $(\ln x)' = \frac{1}{x}$ $(x > 0.)$

(iv) *Exponential & Logarithmic Functions:* $(a^x)' = a^x \ln a$ $(a > 0)$; $(\log_a x)' = \dfrac{1}{x \ln a}$ $(x, a > 0.)$

(v) *Trig. Functions:* $(\sin x)' = \cos x$; $(\cos x)' = -\sin x$; $(\tan x)' = \dfrac{1}{\cos^2 x}$; $(\cot x)' = -\dfrac{1}{\sin^2 x}$.

(vi) *Inverse Trig. Functions:*
$(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$; $(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$; $(\arctan x)' = \dfrac{1}{1+x^2}$; $(\operatorname{arccot} x)' = -\dfrac{1}{1+x^2}$.
Domains and graphs of the trig. and inverse trig. functions and their derivatives?

(b) *Multiplication by a Constant:* If $f(x)$ is a differentiable function, then $(c\,f(x))' = c\,f'(x)$.

(c) *Sum and Difference Rules*: If $f(x)$ and $g(x)$ are differentiable, then their sum and difference are also differentiable: $(f(x) + g(x))' = f'(x) + g'(x)$, and $(f(x) - g(x))' = f'(x) - g'(x)$.

(d) *Product Rule:* If $f(x)$ and $g(x)$ are differentiable, then their product is also differentiable, and $\big(f(x) \cdot g(x)\big)' = f'(x)g(x) + f(x)g'(x)$.

(e) *Quotient Rule:* If $f(x)$ and $g(x)$ are differentiable, and $g(x) \neq 0$ for all x nearby a (or on a given interval (A, B)), then their quotient is also differentiable whose derivative is given by:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

(f) *Chain Rule:* If $F(x) = f(g(x))$ for some differentiable functions $f(x)$ and $g(x)$, then $F(x)$ is also differentiable and its derivative is given by: $F'(x) = f'(g(x)) \cdot g'(x)$.

(g) *Some Miscellaneous DLs.* You must understand what they mean and use them in problems.

- *Power and Chain Rules combined:* $(f^n(x))' = n\,f^{n-1}(x)\,f'(x)$; $\quad (f^c(x))' = c\,(f^{c-1}(x))\,f'(x)$.
- *Reciprocal Rule:* $\left(\dfrac{1}{f(x)}\right)' = \dfrac{-f'(x)}{f^2(x)}$.
- *Logarithmic and Chain Rules combined:* $(\ln f(x))' = \dfrac{f'(x)}{f(x)}$.
- *Exponential and Chain Rules combined:* $(c^{f(x)})' = c^{f(x)} \cdot \ln c \cdot f'(x)$.

(13) *Limit Theorems.* $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$; $\lim_{x \to 0} (1 + x)^{1/x} = e$.

3. Problem Solving Techniques

(1) **How do we find $\lim_{x\to a} f(x)$ when the (finite) LLs fail?**

   (a) If $f(x) = \dfrac{g(x)}{h(x)}$ and "plugging in a" yields $\dfrac{0}{0}$, try factoring polynomials and rationalizing expressions with square roots. The idea is to end up with $(x-a)$ both in numerator and denominator, cancel it, and then again attempt to apply LLs.

   (b) If $f(x)$ is a piecewise-defined function (i.e. given by different formulas on several intervals), try first to find the left-hand and the right-hand limits separately, and then compare them to see if they are equal (or if they exist, for that matter).

   (c) If $f(x)$ is given by a formula involving absolute values, again proceed by finding and comparing the two one-sided limits.

(2) **How do we determine if a function is continuous at a?** By definition of continuity, there are 3 things to check:

   (a) Find $\lim_{x\to a} f(x)$ by following either limits laws or the techniques suggested above. (If it doesn't exist, then the function has no chance of being continuous at a. If it exists but is an infinite limit $\pm\infty$, again the function is not continuous at a; in fact, it has an *infinite discontinuity* at a.)

   (b) Find $f(a)$. If f is not defined at a, then the function is not continuous at a.

   (c) If the above two steps yield two finite numbers, compare them to check if they are equal: $\lim_{x\to a} f(x) = f(a)$. If yes, the function is continuous at a; if not, the function is not continuous at a.

(3) **How do we determine if a function is continuous at a without using the definition of continuity?** We use CLs if applicable. For example, the function $f(x) = \frac{1}{x-3} \cdot \cos x + 6x^3$ is continuous as 2 because all comprising functions (rational, trigonometric and polynomial) are all continuous at 2. However, the function is not continuous at 3. (Why?)

(4) **How do we find $\lim_{x\to \square_1} f(x)$ when infinite limits are involved, but ∞–LLs fail?**

   (a) When finding the limit of a rational function: $\lim_{x\to\pm\infty} \dfrac{P_1(x)}{P_2(x)}$ (here $P_1(x)$ and $P_2(x)$ are polynomials), we know that ∞/∞ doesn't make sense. So, we factor out the highest powers of x from both top and bottom polynomials, cancel, and then apply LLs again. (Note: In the end, all that will matter will be the leading terms of the two polynomials - no other terms will survive the above operations.) Similar ideas apply to any other fractions which involve polynomials and possibly radicals.

   (b) If ∞–LLs produce expressions involving $\infty - \infty$, we know that this doesn't make sense, so we look for a different approach. If polynomials are involved, factoring out the highest power is a good start. If square roots are involved, rationalizing might help. If two or more fractions are involved, putting them under a common denominator to arrive at one single fraction is the first step; then apply other techniques mentioned above.

   (c) If ∞–LLs produce an expression of the type $0 \cdot \infty$, we know that this doesn't make sense, so we look for a different approach. Each example of this type has to be considered individually; most likely, we will end up factoring or rationalizing in search of common things to cancel, and after that we will attempt again to apply LLs.

(5) **How do we sketch graphs of functions $f(x)$?**

(a) If you recognize that the graph of $f(x)$ can be obtained from a graph of a well-known function via horizontal and/or vertical shifts, go for it! If the original (well-known) function already has vertical or horizontal asympotets, make sure to shift them too and indicate that the resulting function has these-and-these asymptotes.

(b) If the graph of $f(x)$ can't be obtained via the above shifts (e.g. $f(x)$ is a complicated function, or you just forgot how to sketch the graph of the "well-known" function), proceed as follows:

(i) First, look for "zeros" of the functions, i.e. for its x-intercepts: try to solve $f(x) = 0$ if possible. If $f(x)$ is a fraction, such solutions will be produced in the numerator. (The denominator will be irrelevant in this step.) It is also good to find the y-intercept, by setting $x = 0$ in $f(x)$.

(ii) Second, look for vertical asymptotes: these will appear where $f(x)$ has an infinite (at least) one-sided limit, i.e. if $\lim\limits_{x\to a^+} f(x) = \pm\infty$ or $\lim\limits_{x\to a^-} f(x) = \pm\infty$; then the vertical line $x = a$ is such an asymptote. If $f(x)$ is a fraction, such solutions will be produced by the roots of the denominator. (The numerator will be irrelevant in this step.)

(iii) Third, look for horizontal asymptotes: these will appear where $f(x)$ has a finite limit when $x \to \pm\infty$, i.e. if $\lim\limits_{x\to\pm\infty} f(x) = L$; then the horizontal line $y = L$ is such an asymptote. If $f(x)$ is a fraction, both numerator and denominator will be involved in this step.

(iv) Finally, draw the vertical and horizontal asymtotes, mark the x-intercepts (and the y-intercept if applicable); draw the function so that it passes through the x- and y-intercepts and respects all asymptotes as found above. Be careful nearby the vertical asymptotes to reflect whether a given one-sided limit is $+\infty$ or $-\infty$, correspondingly. It doesn't hurt to plot several other points in each interval between asymptotes and x-intercepts, to make your graph more precise and make sure you haven't done any silly calculation mistakes in the above steps.

(6) **How do we prove statements about limits using the limit definitions?**

We pray that such a problems is not on the midterm. If this doesn't help, we follow the steps below.

(a) Consider the type of limit you are given: $\lim\limits_{x\to\Box_1} f(x) = \Box_2$ and decide what your goals will be depending on what $\Box_2$ is.

- If $\Box_2 = L$ - a finite number, then your function "wants to be close to this number L"... how close? - ϵ-close. Your goals will be therefore ϵ-goals around L, which can be written in one of the following three equivalent forms:

$$f(x) \text{ is within } \epsilon \text{ of } L \;\Leftrightarrow\; L-\epsilon < f(x) < L+\epsilon \;\Leftrightarrow\; |f(x)-L| < \epsilon$$

- If $\Box_2 = +\infty$, then your function "wants to be close to $+\infty$"... To say "ϵ-close to $+\infty$" doesn't make a whole lot of sense! Instead, we want $f(x)$ to get as large as we please. Hence, our goals will be M-goals, where $M > 0$. Such a goal can be written simply as $f(x) > M$.
- If $\Box_2 = -\infty$, then your function "wants to be close to $-\infty$"... To say "ϵ-close to $-\infty$" doesn't make a whole lot of sense! Instead, we want $f(x)$ to get as small as we please. Hence, our goals will be M-goals, where $M < 0$. Such a goal can be written simply as $f(x) < M$.

(b) Decide next what type of answers you are looking for depending on what $\Box_1$ is.

- If $\square_1 = a$ - a finite number, then x "wants to be close to this number a"... how close? - δ-close. Your answers will be therefore δ-intervals around a, which can be written in one of the following three equivalent forms:

$$x \text{ is within } \delta \text{ of } a \;\Leftrightarrow\; a - \delta < x < a + \delta \;\Leftrightarrow\; |x - a| < \delta.$$

- If $\square_1 = +\infty$, then x "wants to be close to $+\infty$"... To say "δ-close to $+\infty$" doesn't make a whole lot of sense! Instead, we want x to be large enough. Hence, our answers will be M-answers, where $M > 0$, written simply as $x > M$.
- If $\square_1 = -\infty$, then x "wants to be close to $-\infty$"... To say "δ-close to $-\infty$" doesn't make a whole lot of sense! Instead, we want x to be small enough. Hence, our answers will be M-answers, where $M < 0$, written simply as $x < M$.
- **Summary of (ϵ, δ)–Definition Types of Goals and Answers.** Here are all 9 possible types of limits (3 possible goals, and 3 possible answers) for $\lim_{x \to \square_1} f(x) = \square_2$:

limit $\square_2$	goal for $f(x)$	$x \to \square_1$	answer for x
L	ϵ-goal around L	$x \to a$	δ-interval around a
$+\infty$	M-goal, $M > 0$	$x \to +\infty$	N-answer, $N > 0$
$-\infty$	M-goal, $M < 0$	$x \to -\infty$	N-answer, $N < 0$

(c) Put together your goals and your types of answers to see what you are really after. And then proceed either algebraically or graphically.

(i) The algebraic way is more rigorous, but then less insighful. It goes as follows.

Step I: Conjecture your limit.

Step II: Write your goal for your function, e.g. we want $5 - \epsilon < f(x) < 5 + \epsilon$ (here $\square_2 = L = 5$), or $f(x) < M$ (here $\square_2 = -\infty$ and $M < 0$). Next, try to solve these inequalities for x. Keep in mind that sometimes considerations like $x > 0$ or $x < 0$ can help eliminate irrelevant information. Next, put your answer in the form of δ-interval around a, or in the form of $x > M$ or $x < M$, depending on what type of answer you are looking for. (Recall that sometimes when looking for δ, we have to take the smaller of two distances from a in order to ensure that our δ-interval is centered at a.) Make sure you announce what your final δ or M answer is, do NOT forget to state your conclusion: The above steps show that indeed blah-blah ... ($\lim_{x \to \square_1} f(x) = \square_2$)

(ii) The geometric way is less rigorous, but then more intuitive. Start by sketching a graph of your function (you may have to shift graphs of simpler functions or plot several points for your function). Then mark your goal: either an ϵ-strip around L, or an M-region (above or below M depending on whether your limit is $+\infty$ or $-\infty$.)

Mark the portion of the graph which lies inside your goal area. Project this portion onto the x-axis, i.e. mark all x's above which $f(x)$ falls into the goal area. These x's should be grouped in one or more intervals. Make sure you eliminate the irrelevant intervals for x (e.g. if $x \to -\infty$, then intervals with positive x's are irrelevant; if $x \to 7$, then an interval around 2 is probably also irrelevant, but an interval around 7 will be VERY relevant!) You should be left with only one relevant interval I for x.

Using the formula for the function $f(x)$, find precisely what this I is. This usually entails the following calculations: $f(x) = L + \epsilon$ (when the function enters/exits "from above" the ϵ-strip around L); or $f(x) = L - \epsilon$ (when the function enters/exits "from below" the ϵ-strip around L); or $f(x) = M$ (when the function enters/exits the M-goal region). Solve this for x to find your "good interval" I - check with your graph to make sure that what you obtain makes sense.

Finally, translate this into the type of answer you are expected to obtain: $a - \delta < x < a + \delta$, or $x < M$ or $x > M$, and state this answer clearly. (Make sure that all of the above calculations and work with the graph is recorded properly in your solution!) Conclude by stating that therefore blah-blah... ($\lim_{x \to \Box_1} f(x) = \Box_2$)

(7) **How do we find derivatives from the definition?** Read carefully if you are being asked to find a specific derivative $f'(a)$, or the *whole* derivative function $f'(x)$. In each case, you have two choices how to proceed, as listed below.

(a) $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}\cdot$ Here a is a constant and x moves towards a, so we expect that x will disappear and a will remain in the final result for $f'(a)$.

(b) $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}\cdot$ Here a is a constant and h moves towards 0, so we expect that h will disappear and a will remain in the final result for $f'(a)$. This formula is nothing else but formula (a) where x is replaced by $a + h$.

(c) $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}\cdot$ Here x is viewed as a constant and h moves towards 0, so we expect that h will disappear and x will remain in the final result for the derivative function $f'(x)$. This formula is nothing else but formula (b) where a is replaced by x.

(d) One can also find $f'(x)$ by first finding $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$; in the result of this calculation x will disappear and only a will remain; in this final formula replace a by x to obtain a formula for the whole derivative function $f'(x)$.

(8) **How do we find equations of tangent lines?**

(a) First find the corresponding derivative $f'(a)$: this will be the slope of your tangent line.

(b) Next use the point–slope formula for the point $P(a, f(a))$ and for the slope $f'(a)$ from part (a):

$$f'(a) = \frac{y - f(a)}{x - a} \quad \Leftrightarrow \quad y - f(a) = f'(a)(x - a)\cdot \tag{1}$$

Here a and $f(a)$ are constants, and x and y are variables in the equation for your tangent line. Where suitable, multiply through and simplify to obtain a formula of the type:

$$y = f'(a) \cdot x + b.$$

(9) **How do we find all the tangents lines to the graph of $f(x)$ which are parallel to some line $y = mx + b$?**

(a) First find the slope m of the given line; be careful with this since the line may not be given in the standard "line-equation" form as above and you may have to rewrite it.

(b) Find the derivative $f'(x)$.

(c) Next, set $f'(x) = m$, where m is the found slope of the line above, and solve it for x.

(d) Finally, let's say you found several solutions x_1, x_2... etc. What remains is to find the corresponding points on the graph of $f(x)$ through which the wanted tangent lines will pass: $P_1(x_1, f(x_1))$, $P_2(x_2, f(x_2))$, etc. If the problem asks for finding the equation of these tangent lines, well, proceed now with the point-slope formula as before.

(10) **How do we find all tangent lines to the graph of $f(x)$ passing through a point (c, d)?**

(a) First find the derivative $f'(x)$.

(b) Next, set the point-slope formula for the tangent line:

$$f'(a) = \frac{y - f(a)}{x - a}\cdot$$

You will not know at this moment what a is (that's what you want to find in the end), nor what x or y are (these are the variables in the equation of the tangent line).

(c) Substitute the point (c, d) into the above equation ((c, d) is supposed to lie on your tangent line, hence substituting $x = c$ and $y = d$ must work):

$$f'(a) = \frac{d - f(a)}{c - a}.$$

At this point, you must realize that only a is left "unknown" in this equation; everything else must be a number; thus, we can solve this equation for a.

(d) Now that you know what a is (are), find the equation of the corresponding tangent line(s) at $(a, f(a))$. The problem may be asking for less: just find the points of tangency $(a, f(a))$.

(11) **How do we sketch graphs of the derivative function $f'(x)$ given the graph of $f(x)$?**

(a) Find where the given function $f(x)$ is *not* differentiable; at these x's $f'(x)$ will not exist. There are many different reasons for $f'(x)$ not to exist. Here follow some such reasons:

- $f(x)$ is *not* defined at $x = a$. Then we can't even talk about the derivative at $x = a$.
- $f(x)$ is defined at $x = a$ but is *not* continuous there. Then the contrapositive theorem implies that $f(x)$ is *not* differentiable at $x = a$. No matter what type of discontinuity $f(x)$ has at $x = a$, $f'(a)$ will *not* exist. An infinite discontinuity of $f(x)$ (i.e. $f(x)$ has a vertical asymptote $x = a$) usually translates into a vertical asymptote for $f'(x)$ at $x = a$. A jump or removable discontinuity of $f(x)$ usually translates into a jump or removable discontinuity for $f'(x)$. Each case is treated separately to see what happens with $f'(x)$.
- $f(x)$ is defined and continuous at $x = a$, but is *not* "smooth" there, i.e. has a cusp(corner). Usually here either the two one-sided tangents exist at $x = a$ but have different slopes, or there is a vertical tangent at $x = a$. In the former case, this will translate into a jump discontinuity of $f'(x)$; in the latter case, this translates into a vertical asymptote of $f'(x)$.
- $f(x)$ looks "smooth" at $x = a$, but has a vertical tangent there. Again, this will translate into a vertical asymptote of $f'(x)$.

(b) After marking all x's where $f'(x)$ does not exist (including possible vertical asymptotes, etc.), we move on to graphing $f'(x)$ where it exists. First find all places where the tangents to $f(x)$ are horizontal and mark the corresponding 0's on the graph of f'. Next determine the intervals where $f(x)$ increases i.e. has positive tangent slopes, and where $f(x)$ decreases, i.e. has negative tangent slopes. In the former case, $f'(x)$ will be positive, and in latter case, $f'(x)$ will be negative. In each such interval, answer the following two questions: whether the tangent slopes are positive or negative, and whether the tangent slopes themselves are increasing or decreasing. Translate this into the corresponding property of $f'(x)$.

(c) For more precise drawing, in each of the above intervals mark several tangent lines, guestimate their slopes and mark the corresponding points on the graph of $f'(x)$. Connect all these marked points to obtain the graph of $f'(x)$. Don't forget the places where $f'(x)$ was not defined!

(12) **How do we find derivatives using DLs?** If you are given $f(x)$ via one formula and you are not asked to use the definition of derivative, you apply DLs.

(a) First see if you can further simplify the given formula. In particular, try to avoid applying the Quotient Rule whenever possible because it is prone to errors. In practice this mean: try to get rid of denominators by either splitting fractions and then simplifying each fraction separately (see formulas for fraction manipulations below), or by direct cancellation of common stuff in the numerator and denominator, or by moving the denominator into the numerator: e.g. x^3 in the denominator becomes x^{-3} in the numerator.

(b) If you are going to apply the Power Rule, turn all expressions like $\sqrt[n]{x}^m$ into the standard form $x^{\frac{m}{n}}$. Again, such expressions in the denominator should move into the numerator wherever suitable by flipping the sign of the power: $\sqrt[n]{x}^m$ in the denominator becomes $x^{-\frac{m}{n}}$ in the numerator.

(c) Look at your function $f(x)$ to figure out its components, the simpler pieces it is made of, and decide which DL(s) you are going to use. In some cases, you may have to apply several DLs one after the other, so keep good track of your intermediate results, or else your calculations will be untraceable. A good strategy is to name some of the simpler components of $f(x)$, e.g. $g(x)$, $h(x)$, etc. and perform some of the necessary differentiation on these functions on the side and then put back your results together. To reduce errors and to make clear that you do know the DLs, it is always good to write the DL formula in terms of functions at first, e.g.

$$((5x+2)\cdot x^3)' \overset{PR}{=} (5x+2)'\cdot x^3 + (5x+2)\cdot(x^3)' = ...$$

(d) In case your function is given by several formulas on different intervals, you must find the derivative of each such formula on the corresponding interval. In the end, you must compare your results for the left-side and right-side derivative at the "break" points to determine if you function is differentiable there. E.g. if $f(x)$ is defined by two different formulas on $(2,5]\cup(5,8)$, then at the end you must compare $f'_-(5) \overset{?}{=} f'_+(5)$. If yes, then $f'(5)$ also exists; if not, then $f'(5)$ doesn't exist. You final answer for $f'(x)$ is again going to be given by several different formulas on the corresponding intervals.

(13) $F(x)$ **is a composition of several functions. How do we find these functions?** Start with the variable x wherever it appears in the formula for $F(x)$. See what operations are performed on x and record the corresponding functions from right to left. E.g. $F(x) = \sqrt[3]{2\sin x + 5}$:

$$x \overset{f}{\mapsto} \sin x \overset{g}{\mapsto} 2\sin x + 5 \overset{h}{\mapsto} \sqrt[3]{2\sin x + 5}.$$

Name the intermediate expressions by some variables, say, $u = \sin x$, $v = 2\sin x + 5 = 2u + 5$. Thus,

$$x \overset{f}{\mapsto} u = \sin x \overset{g}{\mapsto} v = 2u + 5 \overset{h}{\mapsto} \sqrt[3]{v}.$$

From here we see that $f(x) = \sin x$, $g(u) = 2u + 5$ and $h(v) = \sqrt[3]{v}$. Hence, $F(x) = h(g(f(x))) = (h\circ g\circ f)(x)$, a triple composition. The above method is "from inside-out", i.e. start with the inner functions and move to the outer functions.

(14) **How do we apply the Chain Rule to compositions of three or more functions?** The name "Chain Rule" comes from the expressions "chain reaction". It is a recursive process, but to keep things simple see what happens to the example above:

$$F'(x) = h'(g(f(x)))\cdot g'(f(x))\cdot f'(x).$$

We first find, on the side, all involved derivatives:

$$f'(x) = (\sin x)' = \cos x,\ g'(u) = (2u+5)' = 2,\ h'(v) = (\sqrt[3]{v})' = (v^{1/3})' = \frac{1}{3}v^{\frac{1}{3}-1} = \frac{1}{3}v^{-\frac{2}{3}} = \frac{1}{3v^{\frac{2}{3}}}.$$

We then substitute back: $F'(x) = \dfrac{1}{3v^{\frac{2}{3}}}\cdot 2\cdot\cos x$. Now we recall our substitutions above for $u = \sin x$ and $v = 2\sin x + 5$:

$$F'(x) = \frac{1}{3(2\sin x+5)^{\frac{2}{3}}}\cdot 2\cdot\cos x = \frac{2\cos x}{3(2\sin x+5)^{\frac{2}{3}}} = \frac{2\cos x}{3\sqrt[3]{(2\sin x+5)^2}}.$$

One may also try the opposite way: start with the outermost (or last) function and move backwards to the innermost (first) function. So here goes your favorite **blah**-method:

$$(\sqrt[3]{2\sin x+5})' = \left((2\sin x+5)^{\frac{1}{3}}\right)' = \left((\text{blah})^{\frac{1}{3}}\right)'\cdot(\text{blah})' = \frac{1}{3}(\text{blah})^{\frac{1}{3}-1}\cdot(\text{blah})' =$$

$$= \frac{1}{3}(2\sin x + 5)^{-\frac{2}{3}} \cdot (2\sin x + 5)' = \frac{1}{3}(2\sin x + 5)^{-\frac{2}{3}} \cdot (2\cos x) = \frac{2\cos x}{3\sqrt[3]{(2\sin x + 5)^2}}.$$

(15) **But how do we apply the Chain Rule if there are several occurrences of x?** Several occurrences of x ordinarily means that one chain rule will not suffice, and you will have to apply some other DLs. For example, if $G(x) = \sqrt[3]{2\sin x + 5} \cdot 7^{\tan x}$, then the "biggest" (or last) operation is the multiplication in the middle, so we'll have to apply the product rule to the two functions $F(x) = \sqrt[3]{2\sin x + 5}$ and $H(x) = 7^{\tan x}$:

$$G'(x) = \big(F(x) \cdot H(x)\big)' = F'(x)H(x) + F(x)H'(x).$$

In this case, obviously, to keep our sanity, we'd better differentiate $F(x)$ and $H(x)$ on the side (as if they were two other simpler but independent problems), and then substitute our answers back for $G'(x)$. Thus, good record keeping is of utmost importance here.

(16) **But how do we find the derivative of such an exponential function as $H(x) = 7^{\tan x}$?** Represent $H(x)$ as the composition of two functions: $H(x) = f(g(x))$, where $g(x) = \tan x$ and $f(u) = 7^u$, $u = g(x) = \tan x$. The derivatives of these functions are as follows:

$$g'(x) = (\tan x)' = \frac{1}{\cos^2 x}, \; f'(u) = (7^u)' = 7^u \cdot \ln 7.$$

If you forget the derivative of the tangent function, then proceed with the quotient rule there. The derivative of 7^u was found by applying the Exponential Functions Rule. Now, the Chain Rule implies:

$$H'(x) = f'(g(x)) \cdot g'(x) = (7^u \cdot \ln 7) \cdot \frac{1}{\cos^2 x} = 7^{\tan x} \cdot \ln 7 \frac{1}{\cos^2 x} = \frac{7^{\tan x} \cdot \ln 7}{\cos^2 x}.$$

Again, the people rooting for the **blah**-method will enjoy the following:

$$\big(7^{\tan x}\big)' = \big(7^{\text{blah}}\big)' \cdot (\text{blah})' = 7^{\text{blah}} \ln 7 \cdot (\text{blah})' = 7^{\tan x} \cdot \ln 7 \cdot (\tan x)' = 7^{\tan x} \cdot \ln 7 \cdot \frac{1}{\cos^2 x} = \frac{7^{\tan x} \cdot \ln 7}{\cos^2 x}.$$

(17) **But still, what will be the derivative of $G(x) = \sqrt[3]{2\sin x + 5} \cdot 7^{\tan x}$? Wouldn't that be too complicated?** Yes, the answer would be a heck of an answer if we manage to get to it without making errors, but remember that most of these calculations are algorithmic, i.e. there are rules which can be applied and not too much thinking is required; indeed, you have to be ruthlessly precise and keep your concentration throughout the whole computation process. Besides, who said that life is simple? :) In any case, here it goes: we have found the derivatives of $F(x) = \sqrt[3]{2\sin x + 5}$ and $H(x) = 7^{\tan x}$ above. Because we kept good records of what we did, we can now substitute our results into the product rule for $G(x) = F(x) \cdot H(x)$:

$$G'(x) = F'(x)H(x) + F(x)H'(x) = \frac{2\cos x}{3\sqrt[3]{(2\sin x + 5)^2}} \cdot 7^{\tan x} + \sqrt[3]{2\sin x + 5} \cdot \frac{7^{\tan x} \cdot \ln 7}{\cos^2 x}.$$

(18) **Gosh, what about the derivatives of inverse functions? Do we need to know these too?** A good advise is to have the derivatives of the inverse trigonometric functions on your cheat sheet, AND to understand the graphical ideas behind the derivatives of inverse functions.

If two functions $f(x)$ and $g(y)$ are inverses of each other, then their two compositions must yield the identity function, i.e. $f(g(y)) = y$ and $g(f(x)) = x$. For example, $f(x) = e^x$, and $g(y) = \ln y$.

(a) If we want to draw the graphs of these two functions on the same coordinate system, it will be silly to draw one function in terms of x, and the other in terms of y. (Why? Because then the two graphs will literally coincide, and we won't be able to distinguish between the two functions!! Check out this phenomenon for, say, x^2 and $\sqrt{y}$.) Thus, for graphing purposes, we relabel one of the variables, say, $g(x) = \ln x$, and draw the graphs of $f(x) = e^x$ and $g(x) = \ln x$.

(b) The graphs of two inverse functions are symmetric across the line $y = x$. This means that if $f(x)$ passes through point $(2, 7)$, then $g(x)$ passes through point $(7, 2)$, and conversely. In our particular example above, verify that $f(x)$ passes through points $(0, 1)$ and $(1, e)$, while $g(x)$ passes through points $(1, 0)$ and $(e, 1)$.

(c) The corresponding tangent lines will also be symmetric across the line $y = x$. Thus, the tangent line to the graph of e^x at point $(0, 1)$ will be symmetric to the tangent line to the graph of $\ln x$ at point $(1, 0)$, etc. (Check it on graphically.) This means that the slopes of these two tangent lines will be reciprocal.

(d) To finish off, go back to the original variable $g(y)$, or else there will be a mess. Translating the above into derivatives, we summarize: for any pair of inverse functions $f(x)$ and $g(y)$, their derivatives at the corresponding points are reciprocal to one another:

$$g'(y) = \frac{1}{f'(x)}, \text{ where } y = f(x),\ x = g(y).$$

For example, if $y = f(x) = x^2$ and $x = g(y) = \sqrt{y}$ $(x > 0)$, then we can verify that

$$f'(x) = 2x,\ g'(y) = (\sqrt{y})' = \frac{1}{2\sqrt{y}},$$

$$g'(y) \stackrel{?}{=} \frac{1}{f'(x)} \quad\Leftrightarrow\quad \frac{1}{2\sqrt{y}} \stackrel{?}{=} \frac{1}{2x} \quad\Leftrightarrow\quad \frac{1}{2\sqrt{x^2}} \stackrel{?}{=} \frac{1}{2x} \quad\Leftrightarrow\quad \frac{1}{2x} \stackrel{?}{=} \frac{1}{2x} \text{ Yes.}$$

In particular, for the point $(3, 9)$ on the graph of $f(x) = x^2$: $f'(3) = 6$ while $g'(9) = 1/6$. (We took $g'(9)$, *not* $g'(3)$, since the point $(9, 3)$ lies on the graph of g.) So, indeed, the slopes $f'(3)$ and $g'(9)$ are reciprocal numbers.

(e) **(Extra Stuff. Suitable for Bonus Questions)**. Let's find the derivative of, say, $\operatorname{arccot} x$. We set the pair of inverse functions $y = f(x) = \cot x$, and $x = g(y) = \operatorname{arccot} y$. According to our formula above:

$$g'(y) = \frac{1}{f'(x)}, \text{ i.e. } (\operatorname{arccot} y)' = \frac{1}{(\cot x)'} = \frac{1}{-\frac{1}{\sin^2 x}} = -\sin^2 x.$$

But this is in the **wrong** variable x! And just replacing x by y wouldn't yield the right thing because we have defined x above by $x = \operatorname{arccot} y$. We could plug this in and obtain: $g'(y) = -\sin^2(\operatorname{arccot} y)$, but this is such a horribly inconvenient and uninsightful formula, that we might as well leave the original $g'(y) = (\operatorname{arccot} y)'$.

The solution to this dilemma is as follows, and this is the tricky part. Rewrite somehow $\sin^2 x$ in terms of the original function $\cot x$, and then we'll follow our noses and substitute. OK, but how can we do it? Recall that $\cot x = \frac{\cos x}{\sin x}$, so squaring both sides gives

$$\cot^2 x = \frac{\cos^2 x}{\sin^2 x} = \frac{1 - \sin^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} - 1$$

Solving now for $\sin^2 x$ yields $\sin^2 x = \dfrac{1}{1 + \cot^2 x}$. This is it! Now we have

$$(\operatorname{arccot} y)' = -\sin^2 x = -\frac{1}{1 + \cot^2 x} = -\frac{1}{1 + y^2}.$$

Thus, $(\operatorname{arccot} y)' = -\frac{1}{1+y^2}$, or written in the more traditional notation (now we can relabel y as x if we wish): $(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$.

(19) **How do we use implicit differentiation?** This is used to find tangent lines and their slopes to *curves* in the plane which are *not* graphs of functions (i.e. they violate the vertical line test). Thus, we don't have a function formula to differentiate, but instead an equation for the curve, e.g.

$x^3+x^2y+4y^2=6$. It will be hard, sometimes impossible, to solve such an equation for y, and hence a formula for y may not be available. [2]

(a) We imagine that y is given by such a formula $y=f(x)$ (e.g. $x^3+x^2f(x)+4f(x)^2=6$), and we differentiate (with respect to x) both sides of the given equation, e.g.

$$(x^3+x^2y+4y^2)'=(6)' \;\Rightarrow\; 3x^2+2xy+x^2y'+8y\cdot y'=0$$

Do *not* forget to include y' wherever appropriate, for $y=f(x)$ so that $y'\neq 1$, but $y'=\frac{dy}{dx}=f'(x)$.

(b) Solve the above for y':

$$3x^2+2xy+y'(x^2+8y)=0 \;\Rightarrow\; y'=-\frac{3x^2+2xy}{x^2+8y}.$$

This is the best we can do for y': we have expressed it in terms of x and the original function y.

(c) If we are asked something about derivatives, slopes and tangents at specific places, then we use the above formula for y' and if necessary, the original equation for y. E.g. in our example, find the slope and the equation for the tangent at point $(1,1)$. [3] Now we use the formula for $y'(x)$ and substitute $x=1$, $y=1$:

$$y'(1)=-\frac{3+2}{1+8}=-\frac{5}{9}.$$

Finally, we use the point-slope formula:

$$y'(1)=\frac{y-y(1)}{x-1} \;\Rightarrow\; -\frac{5}{9}=\frac{y-1}{x-1} \;\Rightarrow\; y=-\frac{5}{9}x+\frac{14}{9}.$$

It is always good to check if this is the correct equation for the tangent line: yes, because the slope is $-5/9$, and if we plug in the point $(1,1)$ it works: $1=1$.

(d) Say, we want to find all points on the curve where the tangent to the curve is horizontal. In general, this is not an easy question to answer. Set $y'(x)=0$, and obtain two equations in terms of x and y: the derivative equation and the original equation. Now you are supposed to solve this system of two equations for x and y. In our example, this amounts to:

$$\left|\begin{array}{l} 0=-\frac{3x^2+2xy}{x^2+8y} \\ x^3+x^2y+4y^2=6 \end{array}\right.$$

The first equation yields $0=3x^2+2xy=x(3x+2y)$, i.e. $x=0$ or $y=-\frac{3}{2}x$. Substituting in the second equation: $4y^2=6$ (when $x=0$), or $x^3-\frac{3}{2}x^3+4\frac{9}{4}x^2=6$ (when $y=-\frac{3}{2}x$), i.e. $y=\pm\sqrt{\frac{3}{2}}$, while the second equation is a pain and I won't solve it here. The final answer would have been: the tangent lines to the curve are horizontal at points $(0,\sqrt{\frac{3}{2}})$, $(0,-\sqrt{\frac{3}{2}})$, and at the points yielded by the second case above.

The good news is that if a similar question appears on the exam, the calculations will be easier. The method, however, is outlined above. Note that similarly you can solve all sorts of questions about the tangents to such curves: e.g. find where the tangents are parallel to $y=x$ (set $y'(x)=1$), etc.

(20) **How do we use Logarithmic Differentiation?** Usually, this is used to find the derivatives of functions of the form: $F(x)=f(x)^{g(x)}$. The Chain Rule here doesn't apply easily (why?), so instead we "ln" both sides of the given equation to get rid of the exponent:

$$\ln F(x)=\ln f(x)^{g(x)}=g(x)\ln f(x).$$

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[2] In the particular example, you can indeed solve for y viewing the given equation as a quadratic equation in "y", but believe me, you probably don't want to do that, and you should follow the method of implicit differentiation instead.

[3] Note that this point was obtained by substituting $x=1$ into the original equation: $1+y+4y^2=6$ and obtaining solutions $y=1,-5/4$. Thus, you could have been asked to find instead the tangent line at point $(1,-5/4)$.

In the last step we used the identity $\ln a^b = b \ln a$. Differentiate both sides of the obtained equality: $(\ln F(x))' = (g(x)\ln f(x))'$. On the LHS we use the chain rule, and on the RHS we use the product rule and then the chain rule:

$$\frac{F'(x)}{F(x)} = g'(x)\ln f(x) + g(x)(\ln f(x))'$$
$$\frac{F'(x)}{F(x)} = g'(x)\ln f(x) + g(x)\frac{f'(x)}{f(x)}.$$

Now we recall that we really wanted to get $F'(x)$, so we solve for it:

$$F'(x) = F(x)\cdot\left(g'(x)\ln f(x) + g(x)\frac{f'(x)}{f(x)}\right) = f(x)^{g(x)}\left(g'(x)\ln f(x) + g(x)\frac{f'(x)}{f(x)}\right).$$

The resulting formula is so complicated, that I do not advise you to remember it or even to write it on your cheat sheet. Apply the above procedure to a specific example and work out the whole thing from scratch: it will be easier than to try to apply the above formula.

(21) **How do we use Exponential Differentiation?** As logarithmic differentiation, exponential differentiation can be used to find the derivatives of functions of the form: $F(x) = f(x)^{g(x)}$. This time we represent the function $F(x)$ as an exponent using the formula $e^{\ln y} = y$ where $y = F(x)$:

$$F(x) = e^{\ln F(x)} = e^{\ln f(x)^{g(x)}} = e^{g(x)\ln f(x)}.$$

Now we are in good shape since we can apply the Chain Rule:

$$F'(x) = e^{g(x)\ln f(x)}\cdot(g(x)\ln f(x))'.$$

No wonder that this yields exactly the same (complicated) formula as logarithmic differentiation above. Thus, you have to choose for yourselves which of the two methods you prefer: logarithmic or exponential differentiation, and learn to apply the chosen method on the spot to specific examples.

(22) **What's the use of the Limit Theorem $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ other than applying it to prove that $(\sin x)' = \cos x$?** We already saw how the knowledge of this limit helped us show that

$$\lim_{x\to 0}\frac{\cos x - 1}{x} = 0.$$

One can find many more limits now (to which limit laws are of no use.)

(a) The first idea is to complete, wherever possible, expressions to look like the limit above. E.g.

$$\lim_{x\to 0}\frac{\sin 5x}{3x} = \lim_{x\to 0}\frac{\sin 5x}{5x}\cdot\frac{5}{3} = \frac{5}{3}\lim_{x\to 0}\frac{\sin 5x}{5x}.$$

Now we can substitute $u = 5x$, and note that when $x \to 0$ then $u \to 0$:

$$\frac{5}{3}\lim_{u\to 0}\frac{\sin u}{u} = \frac{5}{3}\cdot 1 = \frac{5}{3}.$$

(b) The second idea is to "force" expressions of the type $(\sin x)/x$ by rewriting our functions, to isolate the "trouble-makers"; to apply to them the above limit theorem, and to apply to the rest the limit laws. E.g. in the following example, LL fail when $x = 0$, so we rewrite the function:

$$\lim_{x\to 0}\frac{\sin x}{x + \tan x} = \lim_{x\to 0}\frac{\sin x}{x + \frac{\sin x}{\cos x}} = \lim_{x\to 0}\frac{\sin x\cos x}{x\cos x + \sin x}.$$

Now, we see that $\cos x$ is no trouble at all since for $x = 0$ it gives $\cos 0 = 1$ and this doesn't mess anything. We can easily isolate the top $\cos x$ by LL for product, but we can't isolate the bottom one so easily. However, if we were to apply the Limit Theorem, each of the two $\sin x$ would require an x "underneath". We achieve this by factoring x in the denominator:

$$\lim_{x\to 0}\cos x\cdot\lim_{x\to 0}\frac{\sin x}{x(\cos x + \frac{\sin x}{x})} = \cos 0\cdot\frac{\lim_{x\to 0}\frac{\sin x}{x}}{\cos 0 + \lim_{x\to 0}\frac{\sin x}{x}} = 1\cdot\frac{1}{1+1} = \frac{1}{2}.$$

(23) **What's the use of the Limit Theorem $\lim_{x\to 0}(1+x)^{1/x}=e$ other than calculating e?** We can find now many more limits, to which LLs failed before. Force the function to look like $(1+x)^{1/x}$:

(a) $\lim_{t\to 0}(1+2t)^{\frac{1}{t}} = \lim_{t\to 0}\left((1+2t)^{\frac{1}{2t}}\right)^2$. Substitute now $x=2t$ and note that when $t\to 0$, then $x\to 0$:

$$\lim_{x\to 0}\left((1+x)^{\frac{1}{x}}\right)^2 = e^2.$$

(b) $\lim_{t\to 0}(1+\frac{t}{3})^{\frac{1}{t}} = \lim_{t\to 0}(1+\frac{t}{3})^{\frac{1}{3t/3}}$. Substitute now $x=\frac{t}{3}$ and note that when $t\to 0$, then $x\to 0$:

$$\lim_{x\to 0}\left((1+x)^{\frac{1}{3x}}\right) = \sqrt[3]{\lim_{x\to 0}(1+x)^{\frac{1}{x}}} = \sqrt[3]{e}.$$

4. Useful Formulas and Miscallaneous Facts

(1) **Quadratic formula**: useful for factoring quadratic polynomials as $a(x-x_1)(x-x_2)$, where x_1 and x_2 are the two roots of the polynomial, and a is the leading coefficient. Useful also for graphing quadratic polynomials: will yield the x-intercepts (or tell you that they don't exist.)

(2) **Rationalizing formula**: $\sqrt{A}-\sqrt{B} = \dfrac{(\sqrt{A}-\sqrt{B})(\sqrt{A}+\sqrt{B})}{\sqrt{A}+\sqrt{B}} = \dfrac{(A-B)}{\sqrt{A}+\sqrt{B}}$.

(3) **Factorization formulas**: $A^2-B^2=(A-B)(A+B)$ and $A^3-B^3=(A-B)(A^2+AB+B^2)$.

(4) **Binomial formulas:** $(A+B)^2=A^2+2AB+B^2$, $(A+B)^3=A^3+3A^2B+3AB^2+B^3$.

(5) **Putting fractions under a common denominator**. The most general formula is as follows: $\dfrac{A}{B}+\dfrac{C}{D}=\dfrac{AD+BC}{BD}$. Yet, it is worth noting that if fractions already share something in their denominators, it will be faster to take this into account, e.g.

$$\frac{2x+1}{x^2}+\frac{x^3}{x(x-1)} = \frac{(2x+1)(x-1)+x\cdot x^3}{x^2(x-1)} = \frac{x^4+2x^2-x-1}{x^2(x-1)}.$$

(6) **Absolute value inequalities**. Note the following expressions which mean the same things:

(a) $|x|<A \;\Leftrightarrow\; -A<x<A$;

(b) $|x-B|<A \;\Leftrightarrow\; -A<x-B<A \;\Leftrightarrow\; B-A<x<B+A \;\Leftrightarrow\; x\in(B-A,B+A)$;

(c) $|3x^2-10|<5 \;\Leftrightarrow\; -5<3x^2-10<5 \;\Leftrightarrow\; 5<3x^2<15 \;\Leftrightarrow\; \sqrt{5/3}<|x|<\sqrt{15/3}$

$\Leftrightarrow\; \sqrt{5/3}<x<\sqrt{15/3}$ (when $x\geq 0$), or $\sqrt{5/3}<-x<\sqrt{15/3}$ (when $x<0$).

As a final answer, the original inequality is satisfied when $x\in(\sqrt{5/3},\sqrt{5})\cup(-\sqrt{5},-\sqrt{5/3})$.

(d) $|2f(x)-7|<0.5 \;\Leftrightarrow\; -0.5<2f(x)-7<0.5 \;\Leftrightarrow\; 3.25<f(x)<3.75 \;\Leftrightarrow\; f(x)\in(3.25,3.75)$.

(7) **Trigonometric functions**: domains of definition, ranges, graphs, periods; where they increase, decrease, values at $x=0,\pi/3,\pi/4$ and so on "prominent" numbers; vertical asymptotes (if any); trigonometric identities; radians versus degrees.

(8) **Exponential and Logarithmic functions**: domains of definition, ranges, graphs; for which bases do these function increase/decrease, c^x and $\log_c x$ are inverse functions of each other, basic identities.

(9) **Manipulations with Fractions**

(a) *Splitting fractions:* $\dfrac{a+b}{c}=\dfrac{a}{c}+\dfrac{b}{c}$; $\dfrac{ab}{cd}=\dfrac{a}{c}\cdot\dfrac{b}{d}$;

(b) **Wrong** *formula:* $\dfrac{a}{b+c}\neq\dfrac{a}{b}+\dfrac{a}{c}$.

(c) *Putting fractions under a common denominator:* $\dfrac{a}{b}+\dfrac{c}{d}=\dfrac{ad+bc}{bd}$.

(d) *When denominators have something in common:* $\frac{a}{be} + \frac{c}{de} = \frac{ad+bc}{bde}$.

(e) *"Fractions over fractions":* $\frac{a}{c} : \frac{b}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$; $\frac{a}{\frac{c}{d}} = \frac{ad}{c}$; $\frac{\frac{a}{b}}{c} = \frac{a}{bc}$.

I cannot conceive of any other operation on fractions! If you think of one, let me know!

(10) **Manipulations with Exponentials and Logarithms**

(a) $a^{b+c} = a^b \cdot a^c$, $\frac{a^b}{a^c} = a^{b-c}$, $(a^b)^c = a^{bc}$, $a^{\frac{b}{c}} = \sqrt[c]{a^b}$, $\frac{1}{\sqrt[c]{a^b}} = \frac{1}{a^{\frac{b}{c}}} = a^{-\frac{b}{c}}$, $a^0 = 1$.

(b) $\ln(e^x) = x$, $e^{\ln x} = x$, $\ln(a^b) = b \ln a$.

(11) **Trigonometric Formulas**

(a) $\sin^2 x + \cos^2 x = 1$;

(b) $\tan x = \frac{\sin x}{\cos x}$; $\cot x = \frac{\cos x}{\sin x}$;

(c) $\sin(\frac{\pi}{2} - x) = \cos x$; $\cos(\frac{\pi}{2} - x) = \sin x$;

(d) $\sin(x+y) = \sin x \cos y + \cos x \sin y$; $\cos(x+y) = \cos x \cos y - \sin x \sin y$.

(e) the values of $\sin x$, $\cos x$, $\tan x$ and $\cot x$ at all "prominent" x's: $0, \pi/6, \pi/4, \pi/3, \pi/2, \pi$, etc.

5. Exercises to Review

(1) **Applying (Finite) Limit Laws**. From §2.3: #2–6

(2) **Finding limits when LL fail at first**.

(a) Factoring. From §2.3: #11–20, Ex.5 on p. 108.

(b) Rationalizing. From §2.3: #21–23,27, Ex.6 on p.108.

(c) Common denominators, simplifying. From §2.3: #25,26,28.

(d) Absolute values. From §2.3: #39–44,47, Ex.7-8 on p.109.

(e) Piecewise defined functions. From §2.3: #45,46,48, Ex.4 on p.108, Ex.9 on p.109.

(f) Applying the Sandwich Theorem. From §2.3: #33–38, Ex.11 on p.111.

(3) **Showing limits via the (ϵ, δ)-definition.**

(a) Finding δ for specific ϵ's. From §2.4: #1–8,13, Example on p.114.

(b) Proving that $\lim_{x\to a} f(x) = L$ via the (ϵ, δ)-definition. From §2.4: #15–28,34, Ex.2,3,4 on pp.117-119.

(c) Proving that $\lim_{x\to a^\pm} f(x) = \pm\infty$ via the (M, δ)-definition. From §2.4: Ex.5 on p.121, #12. For extra practice, try here also #41–42.

(4) **Continuity.**

(a) Determining if functions are continuous via the definition of continuity. From §2.5: #1,10–14, Ex.1-4 on pp.123-126.

(b) Working with graphs to determine if a function is continuous, or what type of discontinuities it has. From §2.5: #3,4,5,6.

(c) Finding one-sided limits to determine continuity: From §2.5: #15–20,35–39,41–43 (graphs here are always helpful.)

(d) Applying Continuity Laws to establish continuity of functions. From §2.5: #21–28, Ex.5 on p.128.

(e) Applying the Intermediate Value Theorem. From §2.5: #44–52, Ex.9 on p.132. (Note: If you are given the interval (A, B), make sure you have one function to work with. If you are not given the interval (A, B), it is your job to find it so that N becomes an intermediate value for the function in question.)

(5) **Summary of limit types**

(a) Using graphs to determine limits. From §2.6: #3–8, Ex.1 on p.137.

(b) Proving that $\lim_{x\to\pm\infty} f(x) = \square_2$ via the limit definitions: From §2.6: #27,28,57–65, Ex.12–13 on p.145.

(c) Proving that $\lim_{x\to\pm\infty} f(x) = \square_2$ via the Infinite Limits Laws and problem solving techniques:

(i) Factoring highest powers in polynomials, rational functions and root functions. From §2.6: #13–22,30–31,33, Ex.3–4 on pp.138-139, Ex.10 on p.142. Check out Theorem 5 on p.138.

(ii) Direct application of infinite limit laws: From §2.6: #26–28, Ex.8–9 on pp.141–142.

(iii) Rationalizing. From §2.6: #23–25,29,35,36, Ex.5 on p.140.

(6) **Finding horizontal and vertical asymptotes and sketching graphs of functions.** From §2.6: #37–44, Ex.11 on p.143, For extra practice: #45–48.

(7) **Sketch the of graph of $f'(x)$ given the graph of** $f(x)$, find where f' does not exist and explain in words why it doesn't exist there, guestimate the slopes of tangents lines. §2.9: #1–13,32–34; Review (p.177-179): #44–46.

(8) **Find $f'(x)$ from the definition of derivative.** §2.8: #13–18, 35–36. §2.9: #21–32. Review (p.177-179): #47,48.

(9) **Find an equation for the tangent line to the graph of $f(x)$ at some point** $(a, f(a))$; find where the tangents lines are horizontal, where they are parallel to a given line $y = mx + b$, which tangents pass through some point (c, d), etc. Here you can use differentiation laws if helpful. §2.7: #5–14. §2.8: #7–10. Review (p.177-179): #37–38,41. §3.1: #44–50,52,54. §3.2: #23-26, 41–42. §3.4: #21–26, #29–30. §3.5: #43–47,51–52. §3.8: #31–32,34.

(10) **Find the derivative $f'(x)$ using Differentiation Laws.** Write all intermediate results. Note that logarithmic and exponential differentiation are simply methods for differentiating, so they can also be used here if applicable. §3.2: #1–22. §3.4: #1–16,27–28. §3.5: #1–42 (be my guest! choose whatever you like). §3.7: #5–20. §3.8: #2–20 (skip #17), #25-30.

(11) **Use implicit differentiation to find the slope and equation of the tangent to the given curve at point (c, d).** Find an equation for the tangent line to the curve. §3.6: #1–30 (anything here is a good example of what may turn out on the test. Learn the method well.)

(12) **Some topics for bonus questions:** piecewise–defined functions and absolute values: §2.9: #46, §3.1: #53–60, §3.5: #48; §3.8: #17. Higher derivatives and Differential Equations: §3.7: #55–64. Logarithmic differentiation: §3.8: #47–49. Limits: §3.4: #35–44, §3.8: #51,52. Miscellaneous problems: §2.9: #43.

6. Cheat Sheet

For the midterm, you are allowed to have a "cheat sheet" - *one–sided* regular 8×11 sheet. You can write whatever you wish there, under the following conditions:

- The whole cheat sheet must be **handwritten by your own hand**! No xeroxing, no copying, (and for that matter, no tearing pages from the textbook and pasting them onto your cheat sheet.)
- Any violation of these rules will disqualify your cheat sheet and may end in disqualifying your midterm. I may decide to randomly check your cheat sheets, so let's play it fair and square. :)
- Don't be a **freakasaurus**! Start studying for the exam several days in advance, and prepare your cheat sheet at least 2 days in advance. This will give you enough time to become familiar with your cheat sheet and be able to use it more efficiently on the exam.