§6.5

15. $p(x) = .01x + 3$; $A = 200$. So in the notation of Figure 8 in the book, the selling price is $B = p(A) = p(200)$. The producers' surplus is

$$\begin{aligned}
&\int_0^{200} B - p(x)\ dx \\
&= \int_0^{200} 5 - .01x - 3\ dx \\
&= \int_0^{200} 2 - .01x\ dx \\
&= 2x - \frac{.01}{2}x^2 \Big|_0^{200} \\
&= 400 - \frac{.01}{2}(200^2) \\
&= 400 - 200 = 200
\end{aligned}$$

dollars.

16. $p(x) = x^2/9 + 1$; $A = 3$. Producers' surplus is

$$\begin{aligned}
&\int_0^{3} p(3) - p(x)\ dx \\
&= \int_0^{3} 2 - \frac{x^2}{9} - 1\ dx \\
&= \int_0^{3} 1 - \frac{x^2}{9}\ dx \\
&= x - \frac{x^3}{27} \Big|_0^{3} \\
&= 3 - \frac{3^3}{27} \\
&= 2
\end{aligned}$$

dollars.

17. $p(x) = x/2 + 7$; $A = 10$.

$$\begin{aligned}
&\int_0^{10} p(10) - p(x)\ dx \\
&= \int_0^{10} 12 - x/2 - 7\ dx \\
&= \int_0^{10} 5 - x/2\ dx \\
&= 5x - x^2/4 \Big|_0^{10}
\end{aligned}$$

$$= 50 - 10^2/4 = 25.$$

18. $p(x) = 1 + \frac{1}{2}\sqrt{x}$; $A = 36$.

$$\begin{aligned}
&\int_0^{36} p(36) - p(x)\ dx \\
&= \int_0^{36} 4 - 1 - \frac{1}{2}\sqrt{x}\ dx \\
&= \int_0^{36} 3 - \frac{1}{2}\sqrt{x}\ dx \\
&= 3x - (\frac{1}{2})(\frac{2}{3})x^{3/2}\Big|_0^{36} \\
&= 3(36) - \frac{1}{3}(36^{3/2}) \\
&= 108 - \frac{1}{3}6^3 = 108 - 72 = 36.
\end{aligned}$$

19. Demand: $p_D(x) = 12 - x/50$; Supply: $p_S(x) = x/20 + 5$. Optimal production is when

$$\begin{aligned}
p_D(x) &= p_S(x) \\
12 - x/50 &= x/20 + 5 \\
7 &= x/50 + x/20 \\
7 &= 7x/100 \\
x &= 100.
\end{aligned}$$

Moreover, $p_D(100) = 12 - 2 = 10$ (and this agrees with $p_S(100) = 10$). So the point of intersection of the curves is $(A, B) = (100, 10)$.

Consumers' surplus:

$$\begin{aligned}
&\int_0^{100} p_D(x) - p_D(100)\ dx \\
&= \int_0^{100} 12 - x/50 - 10\ dx \\
&= \int_0^{100} 2 - x/50\ dx \\
&= 2x - x^2/100\Big|_0^{100} \\
&= 2(100) - 100^2/100 = 100
\end{aligned}$$

dollars. Producers' surplus:

$$\begin{aligned}
&\int_0^{100} p_S(100) - p_S(x)\ dx \\
&= \int_0^{100} 10 - x/20 - 5\ dx \\
&= \int_0^{100} 5 - x/20\ dx
\end{aligned}$$

$$= 5x - x^2/40\Big|_0^{100}$$

$$= 5(100) - 100^2/40 = 500 - 100(\frac{100}{40}) = 500 - 250 = 250$$

dollars.

20. Demand: $p_D(x) = \sqrt{25 - .1x}$. Supply: $p_S(x) = \sqrt{.1x + 9} - 2$. Solving for the intersection point takes some work here, and one must be careful with the logic, as we discuss below. Curves intersect where

$$p_D(x) = p_S(x)$$

$$\sqrt{25 - .1x} = \sqrt{.1x + 9} - 2$$

We can square both sides and the equality still holds:

$$25 - .1x = (.1x + 9) + 4 + 2(-2\sqrt{.1x + 9})$$

Now we'll isolate the square root term:

$$12 - .2x = -4\sqrt{.1x + 9}$$

Now square both sides again:

$$144 + .04x^2 - 24(.2x) = 16(.1x + 9)$$

Now simplify into standard quadratic form:

$$144 - 16(9) + .04x^2 - 24(.2x) - 16(.1x) = 0$$

$$.04x^2 - 6.4x = 0$$

$$x^2 - 25(6.4x) = 0$$

$$x^2 - 160x = 0$$

$$x(x - 160) = 0$$

So we have solutions $x = 0$ and $x = 160$. But the demand curve is strictly decreasing, and the supply curve strictly increasing, so there can't possibly be two intersection points. But notice that all we've shown is:

IF the two curves intersect at x THEN $x = 0$ or $x = 160$.

Because we took squares of both sides (actually twice!), we *haven't* proven the converse ("IF $x = 0$ or $x = 160$, THEN the curves intersect at x"). So we have to check whether either of our contenders does in fact work. So,

$$p_D(0) = \sqrt{25 - .1(0)} = \sqrt{25} = 5,$$

and

$$p_S(0) = \sqrt{.1(0) + 9} - 2 = 3 - 2 = 1.$$

Since $1 \neq 5$, $x = 0$ does not give an intersection point. On the other hand,

$$p_D(160) = \sqrt{25 - .1(160)} = \sqrt{25 - 16} = 3,$$

and

$$p_S(160) = \sqrt{.1(160) + 9} - 2 = \sqrt{25} - 2 = 3,$$

so $x = 160$ *does* give a point of intersection (in fact the unique point of intersection). And that point is $(A, B) = (160, 3)$.

(Note that if $p_S(x) = \sqrt{.1x+9}+2$ instead of $\sqrt{.1x+9}-2$, and $p_D(x)$ is as above, then $(0,5)$ would be the point of intersection instead. You can trace through the algebraic work above and see that one does the same steps; the only difference is the sign on $4\sqrt{.1x+9}$ is positive instead of negative, but this term gets squared, so that sign makes no difference in the end.)

Now on to the consumers' surplus. This is

$$\int_0^{160} p_D(x) - p_D(160)\ dx$$

$$= \int_0^{160} (25-.1x)^{1/2} - 3\ dx$$

$$= (\frac{2}{3})(\frac{1}{-.1})(25-.1x)^{3/2} - 3x\Big|_0^{160}$$

(here the chain rule has been applied (in reverse) to take the antiderivative - we had to divide by the coefficient of $-.1x$).

$$= -3x - \frac{20}{3}(25-.1x)^{3/2}\Big|_0^{160}$$

$$= -3(160) - \frac{20}{3}(25-.1(160))^{3/2} - (-3(0) - \frac{20}{3}(25)^{3/2})$$

$$= -480 - \frac{20}{3}9^{3/2} + \frac{20}{3}5^3$$

$$= -\frac{3(480)}{3} - \frac{20(3^3)}{3} + \frac{20(5^3)}{3}$$

$$= (1/3)(-1440 - 540 + 2500) = 520/3 = 173\frac{1}{3}$$

dollars. Producers' surplus is

$$\int_0^{160} p_S(160) - p_S(x)\ dx$$

$$= \int_0^{160} 3 - \sqrt{.1x+9} + 2\ dx$$

$$= \int_0^{160} 5 - (.1x+9)^{1/2}\ dx$$

$$= 5x - (\frac{2}{3})(\frac{1}{.1})(.1x+9)^{3/2}\Big|_0^{160}$$

$$= 5(160) - \frac{20}{3}(.1(160)+9)^{3/2} + \frac{20}{3}9^{3/2}$$

$$= 800 - \frac{20}{3}(25^{3/2} - 9^{3/2})$$

$$= 800 - \frac{20}{3}(5^3 - 3^3)$$

$$= 800 - \frac{20}{3}(98) = \frac{2400-1960}{3} = \frac{440}{3} = 146\frac{2}{3}$$

dollars.

Note that for problems 37 to 42, there are different possible choices of $f(x)$, n and b, which result in the same Riemann sum. However, there are "better" and "worse" answers, in terms of their optimality in approximating the given sum with an integral. This optimality is between the ease of calculation and the accuracy of the approximation. So, you should choose $f(x)$, n and b so that $\int_a^b f(x)\,dx$ is easy to calculate, but is also a close approximation to the sum (and the sum must of course be a Riemann sum for this integral with n subintervals). Generally the most obvious choice of $f(x)$, b and n is good.

37. A Riemann sum for $\int_a^b f(x)dx$ with n subintervals is of the form

$$\Big[f(x_1) + \ldots + f(x_n)\Big]\frac{b-a}{n} \tag{1}$$

where x_i is in the i^{th} subinterval. Usually the x_i's are chosen with regular spacing; say at the midpoints of their respective subintervals. We need to make the expression produce

$$\Big[(8.25)^3 + (8.75)^3 + (9.25)^3 + (9.75)^3\Big](.5),$$

and $a = 8$ is given. Since $8.25, 8.75, 9.25, 9.75$ are spaced regularly, they probably make good x_i's. Since they are being cubed in the expression, and there are 4 of them, a natural attempt is $f(x) = x^3$, $n = 4$. We want $.5 = (b-a)/n = (b-8)/4$, which holds if $b = 10$. So we have subintervals of width $(b-a)/n = .5$. Then the subintervals are $[8, 8.5]$, $[8.5, 9]$, $[9, 9.5]$ and $[9.5, 10]$. Therefore their midpoints are 8.25, 8.75, 9.25 and 9.75 respectively, so we set these to be x_1, x_2, x_3, x_4. With these choices, we get the desired Riemann sum.

38. The terms

$$\frac{3}{1}, \frac{3}{1.5}, \frac{3}{2}, \ldots, \frac{3}{3.5}$$

are of the form $3/x$, there are 6 of them, and the x-values are spaced at intervals of .5. Moreover, there is a multiplier of (.5) on the desired sum. So a natural guess is $f(x) = 3/x$, $n = 6$. We have $a = 1$, so for $(.5) = (b-1)/6$, we want $b = 4$. This gives subintervals

$$[1, 1.5],\ [1.5, 2], \ldots, [3.5, 4].$$

So we can set x_i to be the left end-point of the i^{th} subinterval, giving $x_1 = 1$, $x_2 = 1.5, \ldots, x_6 = 3.5$. Putting these choices together gives the desired Riemann sum.

39. Here we have 3 terms of the form $x + e^x$, and x-values spaced at intervals of 1. Also the sum multiplier is 1. So setting $f(x) = x + e^x$ and $n = 3$, and with the given $a = 4$, choosing $b = 7$ gives $\Delta x = (b-a)/n = 1$. Then the resulting subintervals are $[4, 5]$, $[5, 6]$, $[6, 7]$. So setting x_i to be the right end-point of the i^{th} subinterval, $x_1 = 5$, $x_2 = 6$ and $x_3 = 7$, as desired. (More explanation is given in problem 37.)

40. Here we have 5 terms of the form $3x^2$, with x-values spaced at intervals of .6, and a sum multiplier of .6. So set $f(x) = 3x^2$, $n = 5$. As $a = 0$, we want $.6 = (b-0)/5$, so $b = 3$. Subintervals are $[0, 0.6]$, $[0.6, 1.2]$, $[1.2, 1.8]$, $[1.8, 2.4]$, $[2.4, 3]$. Then setting x_i to be the midpoint of the i^{th} subinterval, we get $x_1 = .3$, $x_2 = .9$, etc., as desired. (More explanation is given in problem 37.)

41. Since x_i is in the i^{th} subinterval, note that the given sum is a Riemann sum, for $f(x) = (3-x)^2$, $a = 0$, $b = 3$, $n = 100$. Since n is quite large and $(3-x)^2$ does not fluctuate wildly over the given interval (or anywhere), the integral

$$\int_0^3 f(x)dx$$

is a close approximation to the Riemann sum. So the sum is close to

$$\int_0^3 (3-x)^2 dx$$
$$= (-1)\frac{1}{3}(3-x)^3\Big|_0^3$$
$$= -(-\frac{1}{3}3^3) = 9.$$

42. Notice the sum consists of 100 terms of the form $(2x+x^3)(\Delta x)$, with x-values at intervals of $\Delta x = .01$. The x-values range from .01 to 1, so they are at the right end-points of the 100 subintervals of $[0,1]$. So the given sum is a Riemann sum approximation to the integral

$$\int_0^1 2x + x^3\ dx$$
$$= x^2 + \frac{1}{4}x^4\Big|_0^1$$
$$= 1 + (1/4) = 5/4.$$

Since $2x + x^3$ does not fluctuate wildly and there are are large number of subintervals, the Riemann sum is close to the integral's value, so close to 5/4.

Bonus

27. The problem doesn't specify, but let's assume that the daily deposits are all of equal value. Since 1000 dollars is to be deposited per year, the daily deposit is then $1000/365 = 1000\Delta t$ dollars. The problem also doesn't specify how many (and the placement of) leap years occur within the 5 years. This could be 0, 1 or 2. Let's assume it's 0. (Recall that 2100AD is not a leap year.)

(a) They really mean a single day's deposit. The present value of A dollars in t years time, where interest is r per year, compounded continuously, is $P = Ae^{-rt}$. A single days' deposit is $A = 1000\Delta t$, made in t_i years' time. Interset is $4\% = .04$ per year. So the present value of the future deposit is $1000\Delta te^{-.04t_i}$.

(b) Since there are no leap years in our 5 years, there are 5(365) days. So the sum of all the present values of deposits is

$$1000\Delta te^{-.04t_1} + 1000\Delta te^{-.04t_2} + \ldots + 1000\Delta te^{-.04t_{5(365)}}.$$

(c) The sum consists of terms of the form $1000\Delta te^{-.04t}$, where $\Delta t = 1/365$. The t-values also occur in consecutive intervals of width Δt. So the corresponding function is $f(t) = 1000e^{-.04t}$. With the interval $[0,5]$ broken into subintervals of width Δt, t_i is in the i^{th} subinterval. So $[0,5]$ is an appropriate interval.

(d) The integral is
$$\int_0^5 1000e^{-.04t}dt$$
(e) Evaluating, this is
$$\begin{aligned} &\frac{1000}{-.04}e^{-.04t}\Big|_0^5 \\ &= -25000(e^{-.04(5)} - 1) \\ &= 25000(1 - e^{-.2}) \\ &\approx 4532 \end{aligned}$$
dollars.

28. Approximating the continuous income stream with deposits at intervals of Δt, and using the same approach as in problem 27, one obtains a Riemann sum as there. In the limit was $\Delta t \to 0$, the income stream becomes continuous, and one obtains an integral. Similarly to problem 27, this is
$$\begin{aligned} &\int_0^{10} 5000e^{-.05t}dt \\ &= \frac{5000}{-.05}e^{-.05t}\Big|_0^{10} \\ &= -20(5000)(e^{-.05(10)} - 1) \\ &= 100000(1 - e^{-.5}) \\ &\approx 39347 \end{aligned}$$
dollars.