Topics for Review for Final Exam in Calculus 16A

Instructor: Zvezdelina Stankova

Contents

1. Definitions

Understand the following concepts, give examples for each and use them in problems. What is/are:

(1) a *logarithmic function*? How does it relate to exponential functions? The properties of $\ln x$?
(2) *exponential growth* and *exponential decay*? How does the initial size and the rate of growth/decay of the population affect the formula for the population size?
(3) the *half–life* of a radioactive element? How do we calculate it given the decay constant λ? Conversely, how do we use to to calculate λ?
(4) *carbon dating*? How do we use it to determine the age of an artifact?
(5) *continuously compounded interest*?
(6) the *present value* of money to be received in t years?
(7) the *logarithmic derivative* of $f(t)$? How do we use it to find the *relative rate of change* of $f(t)$? What is the *percentage rate of change* of $f(t)$? Which functions have *constant* relative rate of change?
(8) an *antiderivative*? How many antiderivatives does a function $f(x)$ have? What is the connection between antiderivatives of $f(x)$ and the indefinite integral $\int f(x)\,dx$?
(9) the *basic examples* of functions and their antiderivates?
(10) the *linearity properties* of indefinite integrals? How do we use them to compute indefinite integrals?
(11) a *differential equation (DE)?* What do we solve for in a differential equation? What is an *initial condition* of a DE?
(12) a *Riemann sum* of a function $f(x)$ on the interval $[a, b]$? What is Δx? the number n of subintervals? a left–hand, right–hand, or midpoint Riemann sum? How do we choose the sample points x_i in each subinterval? To what does the Riemann sum approach when $n \to \infty$?
(13) the *definite integral* $\displaystyle\int_a^b f(x)\,dx$? What is its connection with the indefinite integral of $f(x)$ and with the net area under $f(x)$?
(14) the *linearity properties* of definite integrals? How do we use them to compute definite integrals?
(15) the *area between two curves*? What is the curves intersect, or what if some of the area appears under the x-axis: do we still take with a positive sign?
(16) the *average* of finitely many numbers? the *average value* of a function $f(x)$ on interval $[a, b]$?
(17) *consumers' surplus*? How do we calculate it?

2. Theorems and Problem Solving Techniques

Understand each of the following theorems and be able to **apply** each theorem appropriately in problems.

(1) **Properties of $\ln x$.** The *natural logarithmic function* $\ln x$ is the inverse of the *natural exponential function* e^x, i.e.

$$\ln(e^x) = x \text{ for all } x, \text{ and } e^{\ln x} = x \text{ for all } x > 0.$$

Hence, the following two equations are equivalent: $\ln a = b$ and $a = e^b$.

(a) $\ln x$ is defined only for $x > 0$.

(b) $\ln 1 = 0$.
(c) $\ln(e^x) = x$ for all x.
(d) $\ln x \nearrow$ for all $x > 0$.
(e) $\ln x > 0$ for all $x > 1$, and $\ln x < 0$ for all $x < 1$.
(f) $\ln x$ is concave up for all $x > 0$.
(g) $(\ln x)' = \dfrac{1}{x}$ for all $x > 0$. More generally, $(\ln |x|)' = \dfrac{1}{x}$ for all $x \neq 0$.
(h) $\ln(xy) = \ln x + \ln y$ for all $x, y > 0$ (ln turns products into sums.)
(i) $\ln(x^y) = y \ln x$ for all y and for all $x > 0$.
(j) $\ln(\dfrac{x}{y}) = \ln x - \ln y$ for all $x, y > 0$ (ln turns quotients into differences.)
(k) $\ln(\dfrac{1}{x}) = -\ln x$ for all $x > 0$.

(2) **General exponential functions.** $(b^x)' = \ln b \, b^x$ for any $b > 0$, for all x.

(3) **Exponential Growth.** If $P(t)$ grows at a rate proportional to its size, i.e. if $P'(t) = kP(t)$ for some growth constant $k > 0$, then $P(t)$ is the function given by:

$$P(t) = P_0 \cdot e^{kt},$$

where $P_0 = P(0)$ is the initial size of the population (i.e. at time $t = 0$).

(4) **Exponential Decay.** If $P(t)$ decays at a rate proportional to its size, i.e. if $P'(t) = kP(t)$ for some decay constant $k < 0$, then $P(t)$ is the function given by:

$$P(t) = P_0 \cdot e^{kt} = P_0 e^{-\lambda t},$$

where $P_0 = P(0)$ is the initial size of the population, and $\lambda = -k > 0$ is a positive constant.

(5) **Half–life.** The half–life t of a radioactive element is calculated by setting $P(t) = \dfrac{1}{2} P_0$, i.e. $P_0 e^{\lambda t} = \dfrac{P_0}{2}$, and solving for λ: $\lambda t = \ln(1/2) = -\ln 2$, i.e. $t = \dfrac{\ln 2}{\lambda}$.

(6) **Carbon Dating.** Given that an artifact contains $r\%$ of the ${}^{14}C$ level in living matter, we determine the age of the artifact by setting the following equation and solving it for t:

$$P(t) = P_0 e^{-0.00012t} = \frac{r}{100} P_0, \Rightarrow \; -0.00012t = \ln(\frac{r}{100}) = \Rightarrow \; t = -\frac{\ln(\frac{r}{100})}{0.00012}.$$

Here we used that $\lambda = -0.00012$ is the decay constant for ${}^{14}C$.

(7) **Interest compounded several times.** If P is the principal amount, r is the yearly interest, and m is the number of times the interest in compounded yearly, the amount of money in t years is given by

$$A(t) = P \left(1 + \frac{r}{m}\right)^{mt}.$$

Note that here 5% interest rate translates into $r = 0.05$ in the above formula.

(8) **Continuously compounded interest.** If P is the principal amount, r is the yearly interest, and the interest is compounded continuously, the amount of money in t years is given by

$$A(t) = Pe^{rt},$$

i.e. the money grows exponentially with growth constant r. Note that here 5% interest rate translates into $r = 0.05$ in the above formula.

(9) **Present Value.** If A amount of money is to be received in t years at interest rate r, the present amount P of A can be calculated by setting $A = A(t) = Pe^{rt}$, and solving for P: $P = Ae^{-rt}$.

(10) **Relative Rate of Change.** The relative rate of change of $f(t)$ is the logarithmic derivative of $f(t)$:

$$\text{relative rate of change} = \frac{d}{dt}\left(\ln f(t)\right) = \frac{f'(t)}{f(t)}.$$

The percentage rate of change of $f(t)$ is the relative rate of change of $f(t)$ expressed as a percentage:

$$\text{percentage rate of change} = \frac{f'(t)}{100\, f(t)}\%.$$

(11) **Antiderivatives.** If $f(x)$ is a continuous function on (a, b), then any two antiderivatives of $g(x)$ differ by a constant, i.e. if $F(x)$ and $G(x)$ are two antiderivatives of $g(x)$, then $G(x) = F(x) + C$ for some constant C. Consequently, to find *all* antiderivatives of $f(x)$, it suffices to find one such antiderivative $F(x)$, and then add C:

$$\int f(x)\, dx = F(x) + C, \text{ for } C \in \mathbb{R}.$$

(12) **Everywhere zero derivative.** If $F'(x) = 0$ for all $x \in (a.b)$, then $F(x)$ is a constant function on (a, b), i.e. $F(x) = C$ for some constant C.

(13) **Basic examples of antiderivatives.**

$f'(x)$	$f(x)$	Check
k	$kx + C$	$(kx + C)' = k$
$x^n, n \neq -1$	$\frac{x^{n+1}}{n+1} + C$	$\left(\frac{x^{n+1}}{n+1} + C\right)' = x^n$
$x^{-1} = \frac{1}{x}$	$\ln\lvert x\rvert + C$	$(\ln\lvert x\rvert + C)' = \frac{1}{x}$
e^x	$e^x + C$	$(e^x)' = e^x$
e^{rx}	$\frac{e^{rx}}{r} + C$	$\left(\frac{e^r x}{r} + C\right)' = e^{rx}$
$a^x,\ a > 0$	$\frac{a^x}{\ln\, a} + C$	$\left(\frac{a^x}{\ln\, a} + C\right)' = a^x$
$(ax + b)^n,\ n \neq -1$	$\frac{(ax+b)^{n+1}}{a(n+1)} + C$	$\left(\frac{(ax+b)^{n+1}}{a(n+1)} + C\right)' = (ax + b)^n$
$\frac{1}{ax+b}$	$\frac{\ln\lvert ax+b\rvert}{a} + C$	$\left(\frac{\ln\lvert ax+b\rvert}{a} + C\right)' = \frac{1}{ax+b}$

(14) **Linearity Properties of Indefinite Integrals (IL's).**

(a) The integral of a sum is the sum of the integrals:

$$\int (f(x) + g(x))dx = \int f(x)\, dx + \int g(x)\, dx.$$

(b) The integral of a difference is the difference of the integrals:

$$\int (f(x) - g(x))dx = \int f(x)\, dx - \int g(x)\, dx.$$

(c) Constants jump in front of integrals:

$$\int c\, f(x)dx = c \int f(x)\, dx.$$

Warning: Integrals of products are not equal to products of integrals:

$$\int (f(x) \cdot g(x))dx \neq \left(\int f(x)\, dx\right) \cdot \left(\int g(x)\, dx\right).$$

(15) **Differential Equations (DEs).** Given a differential equation $f' = g(x)$ with initial condition $f(a) = b$, we solve for $f(x)$. First, find an antiderivative $G(x)$ for $g(x)$ and set $f(x) = G(x) + C$. Next, plug in the initial condition: $f(a) = b = G(a) + C$ and solve for C. Finally, list the function $f(x) = G(x) + C$ for the newly–found $G(x)$ and C.

(16) **Velocity–distance problems.** Given the velocity $v(t)$ of an object, to find the distance $s(t)$ travelled between times $t = a$ and $t = b$, set up the DE: $s'(t) = v(t)$ and hence $s(t) = \int v(t)\,dt$. Find an antiderivative $G(t)$ of $v(t)$ and set $s(t) = G(t) + C$. Finally, subtract: $s(b) - s(a) = G(b) - G(a)$.

(17) **Riemann sums.** Let $f(x)$ be a continuous function on $[a, b]$. To set up the nth Riemann sum of $f(x)$, we divide the interval $[a, b]$ into n subintervals of length $\Delta x = (b - a)/n$. We choose some x_i in each subinterval: if asked for left–endpoint (right–endpoint, midpoint) approximation, choose x_i to be the left–endpoint (right–endpoint, midpoint) of the ith subinterval. Add up the areas of the resulting rectangles to form the desired Riemann sum:

$$S_n = \Delta x \cdot (f(x_1) + f(x_2) + \cdots + f(x_n)) .$$

When $n \to \infty$, the Riemann sum approaches the "net area" under the graph of $f(x)$:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \Delta x\,(f(x_1) + f(x_2) + \cdots + f(x_n)) = \text{net area under the graph of } f(x) = \int_a^b f(x)\,dx.$$

Warning: Areas are always positive. Net areas, definite integrals and Riemann sums may be positive or negative!

(18) **Fundamental Theorem of Calculus.** Let $f(x)$ be a continuous function on $[a, b]$. Then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a),$$

where $F(x)$ is one antiderivative of $f(x)$, i.e. $F'(x) = f(x)$.

(19) **Linearity Properties of Definite Integrals.**

(a) The integral of a sum is the sum of the integrals:

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

(b) The integral of a difference is the difference of the integrals:

$$\int_a^b (f(x) - g(x))dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$$

(c) Constants jump in front of integrals:

$$\int_a^b c\,f(x)dx = c\int_a^b f(x)\,dx.$$

(20) **Area between curves.** Let $f(x)$ and $g(x)$ be two continuous functions on $[a, b]$ such that $f(x) \geq g(x)$, i.e. $f(x)$ is above $g(x)$ on $[a, b]$. Then the area between the two curves is

$$A = \int_a^b \big(f(x) - g(x)\big)dx.$$

If the two curves intersect at several points, we partition the interval $[a, b]$ into several subintervals $[a_1, a_2], [a_2, a_3], [a_3, a_4], \ldots$ so that on each interval one of the two functions is entirely above the other function, and then we add up the corresponding areas:

$$A = \int_{a_1}^{a_2} \big(f(x) - g(x)\big)dx + \int_{a_2}^{a_3} \big(g(x) - f(x)\big)dx + \int_{a_3}^{a_4} \big(f(x) - g(x)\big)dx + \cdots ,$$

where $f(x) \geq g(x)$ on $[a_1, a_2]$, $g(x) \geq f(x)$ on $[a_2, a_3]$, $f(x) \geq g(x)$ on $[a_3, a_4]$, etc. Thus, each piece of area between $f(x)$ and $g(x)$ ends up being taken with a **positive** sign.

(21) **Intersection points.** To find the intersection points of two curves $f(x)$ and $g(x)$, set $f(x) = g(x)$ and solve for x. Finally, plug in the found values for x into $f(x)$ (or $g(x)$) to arrive at the actual *points* of intersection $(x, f(x))$.

(22) **Average Values.** Given several numbers $x_1, x_2, ..., x_n$, their (arithmetic) average is calculated by $(x_1 + x_2 + \cdots + x_n)/n$. Given a function $f(x)$ on interval $[a, b]$, the *average value of* $f(x)$ on $[a, b]$ is calculated by

$$\text{average value of } f(x) = \frac{\int_a^b f(x)\,dx}{b-a}.$$

(23) **Area of a Circle.** The area of a circle of radius r is πr^2.

(24) **Volume of a Solid of Revolution.** The volume of a solid obtained by rotating the graph of $f(x)$ about the x–axis over $[a, b]$ is give by:

$$\int_a^b \pi f^2(x)\,dx.$$

(25) **Volume of a Sphere.** The volume of a sphere of radius R is $\frac{4}{3}\pi R^3$.

(26) **Volume of a Pyramid.** The volume of a pyramid is given by $\frac{1}{3}(\text{base area} \cdot \text{height})$. The volume of a (right circular) cone of radius r and height h is given by $\frac{1}{3}\pi r^2 h$.

(27) **Consumers' Surplus.** Given a demand curve $p = f(x)$, the consumers' surplus is calculated by

$$\int_0^A (f(x) - f(A))\,dx,$$

where the quantity demanded is A, and the price asked is $f(A) = B$.

(28) **Formulas for Sums.**

$$\begin{aligned}
1 + 2 + 3 + \cdots + n &= \frac{n(n+1)}{2}; \\
1^2 + 2^2 + 3^2 + \cdots + n^2 &= \frac{n(n+1)(2n+1)}{6}; \\
1^3 + 2^3 + 3^3 + \cdots + n^3 &= \left(\frac{n(n+1)}{2}\right)^2; \\
a + ar + ar^2 + ar^3 + \cdots + ar^n &= a\frac{1 - r^{n+1}}{1 - r}, \quad r \neq 1.
\end{aligned}$$

3. Exercises to Review

Review **all** class, homework and quiz problems and solutions. These should be sufficient to do well on the exam.

4. Cheat Sheet

For the exam, you are allowed to have a "cheat sheet" - *one page* of a regular 8×11 sheet. You can write whatever you wish there, under the following conditions:

- The whole cheat sheet must be **handwritten by your own hand**! No xeroxing, no copying, (and for that matter, no tearing pages from the textbook and pasting them onto your cheat sheet.)
- Any violation of these rules will disqualify your cheat sheet and may end in disqualifying your exam. I may decide to randomly check your cheat sheets, so let's play it fair and square. :)
- Don't be a **freakasaurus**! Start studying for the exam several days in advance, and prepare your cheat sheet at least 2 days in advance. This will give you enough time to become familiar with your cheat sheet and be able to use it more efficiently on the exam.