Topics for Review for Midterm II in Calculus 16A

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Contents

1. Definitions

Understand the following concepts, give examples for each and use them in problems. What is/are:

(1) an *increasing* function; a *decreasing* function?
(2) a *global/absolute minimum* and a *global/absolute maximum* of a function?
(3) a *local/relative minimum* and a *local/relative maximum* of a function?
(4) a *global extremum* and a *local extremum* of a function? Is a local extremum necessarily a global extremum? examples? Is a global extremum necessarily a local extremum? examples? Can an endpoint be a local extremum? a global extremum?
(5) a *potential* and a *realized* local extremum?
(6) a *concave up* (*concave down*) function?
(7) an *inflection* point? How do we locate all of them?
(8) a *table* for f, f' and f'' and how do we use it to graph $f(x)$? What other calculations, besides those in the table, do we need to do before graphing $f(x)$? (e.g. think of calculating x– and y–intercepts, extremal and inflection **y-**values of $f(x)$.)
(9) an *optimization problem*? What is the strategy for solving optimization problems? What is an *objective* equation and a *contraint* equation?
(10) the *area* and *perimeter* of a figure? Basic examples of figures.
(11) the *inventory* cost? *ordering* cost? *carrying* cost? *production run* cost? the *average inventory level*? How do we calculate all these?
(12) the *marginal* cost? the connection between minimal cost and marginal cost? How do we calculate marginal cost?
(13) the *revenue* in a *non-monopoly* and in a *monopoly* situation? How do we calculate the revenue? What is the *marginal* revenue and how do we calculate it? What is the connection between maximal revenue and marginal revenue?
(14) the *demand* curve? Is it used to compute the cost or the revenue?
(15) the *profit* and how do we find it? the *marginal* profit and how does it relate to marginal cost and marginal revenue? What is the notation for "rate of change of profit with respect to time"?
(16) a *composition* of two functions? What are the "inside" and the "outside" functions? How do we use them to compute the derivative of a composite function?
(17) *implicit* differentiation? In what problems do we use it?
(18) *related rates*? What information do we need to compute $y'(t)$ in problems involving related rates?
(19) *exponential* function? Why is e^x a very special exponential function? What is the number e? the properties of e? of e^x? of b^x?
(20) a *differential equation*? What do we solve for in a differential equation?

2. Theorems

Be able to **write** what each of the following theorems says. Be sure to understand and distinguish between the conditions (hypothesis) of each theorem and its conclusion. Be prepared to **give** examples for each theorem, and most importantly, to **apply** each theorem appropriately in problems.

(1) **Theorem I.(Differentiable $\Rightarrow$ Continuous.)** If $f(x)$ is differentiable at a, then $f(x)$ is continuous at a. If $f(x)$ is differentiable everywhere on its domain, then it is also continuous everywhere on its domain.

(2) **Contrapositive Theorem II. (Non-differentiable $\Rightarrow$ Non-continuous.)** If $f(x)$ is *not* continuous at a, then $f(x)$ is *not* differentiable at a.

(3) **Converse Statement is False!** Continuity does not guarantee differentiability. Counterexample?

(4) **Differentiation Laws (DLs).**

- (a) *Constant Functions:* $(c)' = 0$ for any constant c.
- (b) *Power Rule:* $(x^c)' = c\,x^{c-1}$ for any constant c.
- (c) *Natural Exponential Function:* $(e^x)' = e^x$.
- (d) *Multiplication by a Constant:* If $f(x)$ is a differentiable function, then $(c\,f(x))' = c\,f'(x)$.
- (e) *Sum and Difference Rules*: If $f(x)$ and $g(x)$ are differentiable, then their sum and difference are also differentiable: $(f(x)+g(x))' = f'(x)+g'(x)$, and $(f(x)-g(x))' = f'(x)-g'(x)$.
- (f) *Product Rule:* If $f(x)$ and $g(x)$ are differentiable, then their product is also differentiable, and $\big(f(x)\cdot g(x)\big)' = f'(x)g(x)+f(x)g'(x)$.
- (g) *Quotient Rule:* If $f(x)$ and $g(x)$ are differentiable, and $g(x) \neq 0$ for all x nearby a (or on a given interval (A,B)), then their quotient is also differentiable whose derivative is given by:
$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}.$$
- (h) *Chain Rule:* If $F(x) = f(g(x))$ for some differentiable functions $f(x)$ and $g(x)$, then $F(x)$ is also differentiable and its derivative is given by: $F'(x) = f'(g(x))\cdot g'(x)$.
- (i) *Power and Chain Rules combined:* $(f^n(x))' = n\,f^{n-1}(x)\,f'(x)$; $\quad (f^c(x))' = c\,(f^{c-1}(x))\,f'(x)$.
- (j) *Exponential and Chain Rules combined:* $(e^{f(x)})' = e^{f(x)}\cdot\cdot f'(x)$.

(5) **Theorem for Local Extrema.** If $f(x_0)$ is a local min/max and $f'(x_0)$ exists, then $f'(x_0) = 0$.

(6) **Contrapositive Statement.** If $f'(x_0) \neq 0$, then $f(x)$ cannot have a local extremum at x_0.

(7) **Converse Statement is False.** If $f'(x_0) = 0$, this does not guarantee that $f(x)$ has a local extremum at x_0. Why? Counterexample?

(8) **First Derivative Test.** Does it always work? Examples.

(9) **Second Derivative Test.** Does it always work? Examples.

(10) **Concavity Test.** Does it always work? Examples.

(11) **Inflection Point Test.** Does it always work? Examples.

(12) **Average Inventory Level.** If x is the size of each order throughout the year in a store, then the average inventory level (average number of items in store at any time) is $x/2$ and hence the carrying cost is $a\cdot x/2$, where a is the carrying cost per item per year.

(13) **Defining e via a Limit.** $\displaystyle\lim_{h\to 0}\frac{e^h-1}{h} = 0.$

(14) **Solutions to a DE.** All functions $f(x)$ that satisfy the differential equation $f'(x) = kf(x)$ are of the form $f(x) = Ce^{kx}$ for some constant C.

3. Problem Solving Techniques

(1) **How do we sketch graphs of functions $f(x)$?**

(a) Find where $f(x)$ is defined, where it is continuous and where it is differentiable.

(b) Find where $f(x)$ is increasing and where it is decreasing, using $f'(x)$ and the relevant test(s). (Start your "f, f', f''"–table.)

(c) Find the local and global extrema (if any) of $f(x)$, using $f'(x)$, $f''(x)$ and the relevant test(s).

(d) Find where the function is concave-up and concave-down, and all inflection points, using $f''(x)$ and the relevant test(s).

(e) Find the vertical asymptotes (if any). They will appear where $f(x)$ has an infinite (at least) one-sided limit, i.e. if $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$; then the vertical line $x = a$ is such an asymptote. If $f(x)$ is a fraction, such solutions will be produced by the roots of the denominator.

(f) If possible, find the roots of the function, i.e. its x-intercepts: try to solve $f(x) = 0$ if possible. If $f(x)$ is a fraction, such solutions will be produced in the numerator. (The denominator will be irrelevant in this step.) It is also good to find the y-intercept, by setting $x = 0$ in $f(x)$.

(g) On the x-axis, mark all interesting points which you found above, mark the intervals where $f(x)$ is increasing or decreasing (or constant, for that matter). Mark the x-intercepts and the y-intercept (if any). Mark any local and global extrema you found. Mark any inflection points you found.

(h) Draw the vertical and horizontal asymptotes (if any). Make a mental picture of what is happening nearby each of these asymptotes. Be careful nearby the vertical asymptotes to reflect whether a given one-sided limit is $+\infty$ or $-\infty$.

(i) At this moment, if you are not sure how the graph looks in some intervals, plot a few other points there.

(j) All that is left is to connect the points you plotted respecting the properties you found above: domain of definition, increase/decrease, extrema, concavity and inflection points, dis-continuity and non-differentiability, asymptotes. $\square$

(2) **Summary of Derivative Tests**

#	Test	Uses	Tells us about $f(x)$	Remarks
1	Monotone Test	$f'(x) > 0$ or $f'(x) < 0$	$f(x)$ is increasing or decreasing	bullet–proof
2	1st Derivative Test	$f'(x_0) = 0$, and $f'(x)$ changes sign at x_0	potential local min/max at x_0, realized local min/max at x_0	bullet–proof
3	2nd Derivative Test	$f'(x_0) = 0$, and $f''(x_0) > 0$ or $f''(x_0) < 0$	potential local min/max at x_0, realized local min/max at x_0	fails if $f'(x_0) = 0 = f''(x_0)$
4	Concavity Test	$f''(x) > 0$ or $f''(x) < 0$	$f(x)$ is concave up or down	bullet–proof
5	Inflection Pt Test	$f''(x_0) = 0$, and $f''(x)$ changes sign at x_0	$f(x)$ has inflection point at x_0	$f''(x_0) = 0$ alone does not imply inflection pt at x_0

(3) **How do we solve optimization problems?** Optimization problems come in great varieties of themes, and each problem requires individual consideration. Therefore, there is no uniform way of solving these problems. Below we describe the general *strategy* for solving optimization problems.

(a) Let the optimization problem we are considering be named Problem 1.

(b) Translate Problem 1 into a mathematical Problem 2. For this, one has to understand very well the original Problem 1, and use whatever means necessary for the mathematical translation. Usually, one ends up with a function $f(x)$, and is being asked to find its global or local extrema.

(c) Solve the mathematical Problem 2 using the techniques learned in Calculus I.

(d) Translate the math answer in Problem 2 into a practical answer in the original Problem 1. $\square$

(4) **What is the "Distance Formula" and how do we use it?**

(a) The *Distance Formula* (DF) tells us the distance between two known points, i.e. we know their coordinates. Thus, if $A(x_1, y_1)$ and $B(x_2, y_2)$, the distance between A and B will be

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

(b) The *Distance Formula* often comes in handy in Optimization Problems. Say, for example, we are on a deck of a ship. The front portion of the deck is in the shape of a parabola $y = x^2$. Suppose a person is drowning in the water at location $(18, 0)$. Where should we stand on the deck of the ship and throw a life-saving device?

Obviously, we should stand at the place on the deck which is closest to the drowning person. OK, we don't know where this place will be, but for sure it must be on the edge of the deck! Thus, if we stand somewhere on the edge of the deck, we are standing at some point (x, x^2) - because of the parabolic shape of the deck. The distance between us and the drowning person is given by:

$$d(x) = \sqrt{(x - 18)^2 + (x^2 - 0)^2} = \sqrt{(x - 18)^2 + x^4}.$$

Therefore, all we have to do is find the minimum of this function and where this minimum occurs!

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However, an important simplification is to notice that $d(x)$ will be minimal exactly when its square $d^2(x)$ is minimal. This allows us to get rid of the uncomfortable square in this case and deal with much simpler derivatives. Thus, we set out to find the global minimum of $f(x) = (x - 18)^2 + x^4$.

$$f'(x) = 2(x - 18) + 4x^3 = 2(2x^3 + x - 18).$$

We are interested where $f'(x) = 0$, i.e. $2x^3 + x - 18 = 0$. Let's hope that we won't have to solve such a complicated equation on the test, but for now: notice that $x = 2$ is a root if this equation, so that $(x-2)$ must factor out. Indeed: $2x^3 + x - 18 = (x-2)(2x^2 + 4x + 9)$ (check it out!) The quadratic factor $2x^2 + 4x + 9$ has no zeros, as the quadratic formula yields a negative number under a square root (check this too!). Thus, the only critical point of $f(x)$ is at $x = 2$.

To determine if $f(2)$ is indeed a local extremum, we could find $f''(x)$: $f''(x) = 6x^2 + 1 > 0$ always. In particular, $f(2) > 0$, and indeed $f(x)$ has a local minimum at $x = 2$. Moreover, since $2x^2 + 4x + 9$ is always positive – it is a concave up parabola with no zeros!! – the sign of $f'(x) = (x - 2)(2x^2 + 4x + 9)$ is captured by $x - 2$. Hence, $f'(x) > 0$ for $x > 2$, and $f'(x) < 2$ for $x < 2$, which again confirms that $f(x)$ has a local (and global!) minimum at $x = 2$.

Correspondingly, the distance $d(x)$ is minimum when $x = 2$. This is what we were after: we should position ourselves at point $(2, 4)$ on the deck so that we are closest to the drowning person. The actual distance will be $\sqrt{(18-2)^2 + (4-0)^2} = \sqrt{256+16} = \sqrt{272} \sim 16.5$m. $\square$

(5) **How do we use area formulas in optimization problems?**

(a) First, let's see what area formulas we know. The most basic ones are:
- Area of a square with side x is x^2.
- Area of a rectangle with sides x and y is $x \cdot y$.
- Area of a triangle with base x and height h is $\frac{x \cdot h}{2}$.
- Area of a circle with radius r is πr^2.

(b) It could happen that we also need formulas for the *perimeter* of the above basic figures. (If we take a walk along the sides/edges of a figure, the total length of the walk is called *the perimeter* of the figure.)
- Perimeter of a square with side x is $4x$.
- Perimeter of a rectangle with sides x and y is $2x + 2y$.
- Perimeter of a triangle with sides x, y and z is $x + y + z$.
- Perimeter of a circle with radius r is $2\pi r$.

(c) See if your optimization problem is asking for a maximum or minimum area. (The word "area" may not be mentioned, but the problem could still boil down to an "area" of some sort.) Find the applicable area formula(s), and create a function $f(x)$ describing the given problem.

(d) For example, say we want to make flower and vegetable gardens up in the Berkeley Hills, and put up a fence around them to prevent deer from eating by mistake all our carnations and juicy Bulgarian tomatoes. We want a circular flower garden, and a square vegetable garden. Unfortunately, we aren't rich, yet the fence costs about \$50 per yard! After long soul (and pocket) searching we decide that we can afford at most \$1000 for the fence. What sizes should we choose for the flower and vegetable gardens so that the total garden area is as large as possible?

- The whole story about the price of the fence simply means that we'll have at most a 20 yard fence total for both gardens. If we want to encompass the largest amount of ground, then we must use the longest possible fence - this much is obvious without any higher mathematics.
- The problem is talking about shapes and areas of gardens, hence let's see what area formulas will be relevant: the area of a square x^2 and the area of a circle πr^2. However, the problem is also talking about fences, hence the useful perimeter formulas will be $4x$ for a square and $2\pi r$ for a circle.
- Let's assign letters to the hypothetical sizes of the two gardens: say, the square garden will have size x yards and the circular garden will have radius r yards. The total area of the two gardens will be: $x^2 + \pi r^2$. We want to maximize this quantity.
- The fence is, as we decided above, 20 yards total. The total perimeter of the two gardens will be $4x + 2\pi r$. Hence we want $20 = 4x + 2\pi r$. This means that the size x and the radius r are not completely independent of each other! For example, we can solve for x:
$$x = \frac{20 - 2\pi r}{4} = 5 - \frac{\pi r}{2}.$$

- So, let's substitute x is the formula for the total area above: $(5-\frac{\pi r}{2})^2+\pi r^2$. Simplifying:
$$f(r)=\left(\frac{\pi^2}{4}+\pi\right)r^2-5\pi r+25.$$
Note that this is a function in *one variable only*, namely r! And not a complicated function at all - just a quadratic polynomial. We want to maximize it.
- $f'(r)=\dfrac{\pi^2+4\pi}{2}r-5\pi$. Thus, $f'(r)=0$ if $r=\frac{10}{\pi+4}$ (Why? Do it yourself.) To make sure that this is indeed a local extremum, we check $f''(r)=\frac{\pi^2+4\pi}{2}$, which is always positive. Hence this is a local (and global in this case) minimum. But we wanted the global **maximum**! Our parabola decreases until it hits the above local minimum, and then increases. Hence its global maximum will be attained at the ends of the interval.
- What interval are we talking about? A parabola is defined everywhere! However, the real situation poses some restrictions. For example, we can't have negative x or r - they must both be ≥ 0. Since $x=5-\frac{\pi r}{2}\geq 0$, we discover that $r\leq 10/\pi$, thus the interval of r is $[0,10/\pi]$.
- Thus, $f(x)$ will attain its maximum at one of the ends of $[0,10/\pi]$: $f(0)=25$, and $f(10/\pi)=100/\pi\sim 31.8$. The second case obviously wins, but what does it mean? To have maximal total area we must have $r=10/\pi$ and $x=5-\frac{\pi r}{2}=5-\frac{\pi 10}{2\pi}=0$. In real life, our answer means that if we really want to maximize the used area, we will have no square garden, and the whole land (and fence) should go into a circular flower garden of radius $10/\pi\sim 3.18$ yards.
- A further discussion is due here. We see that if we want to have some vegetable garden too, we have to change somehow our requirements in the original problem! We simply cannot have BOTH 20 yards of fence AND maximal total area, and still expect that we will have a square vegetable garden! Under these conditions, the circular garden will win no matter what, and it will use up all the ground! $\square$
- Suppose instead that we had already bought the 20 yard fence, and wanted to use all of it (well, we won't throw it away!), but we had a shortage of space, so we wanted to use as little space for both gardens as possible. The set-up of the function $f(r)$ will be exactly the same as above, except that we will be looking for its minimum.
We already found where this happens: at $r=\frac{10}{\pi+4}\sim 1.4$, and correspondingly, $x=5-\frac{\pi r}{2}=\frac{20}{\pi+4}\sim 2.8$. These will be two jolly small gardens: a circular garden of radius about 1.4 yards, and a square garden of size about 2.8 yards... But we get what we asked for: the total fence used will be exactly 20 yards, and the total area used will be as small as possible. $\square$

(6) **How do we sketch graphs of the derivative function $f'(x)$ given the graph of $f(x)$?**

(a) Find where the given function $f(x)$ is *not* differentiable; at these x's $f'(x)$ will not exist. There are many different reasons for $f'(x)$ not to exist. Here follow some such reasons:
- $f(x)$ is *not* defined at $x=a$. Then we can't even talk about the derivative at $x=a$.
- $f(x)$ is defined at $x=a$ but is *not* continuous there. Then the contrapositive theorem implies that $f(x)$ is *not* differentiable at $x=a$. No matter what type of discontinuity $f(x)$ has at $x=a$, $f'(a)$ will *not* exist. An infinite discontinuity of $f(x)$ (i.e. $f(x)$ has a vertical asymptote $x=a$) usually translates into a vertical asymptote for $f'(x)$ at $x=a$. A jump or removable discontinuity of $f(x)$ usually translates into a jump or removable discontinuity for $f'(x)$. Each case is treated separately to see what happens with $f'(x)$.
- $f(x)$ is defined and continuous at $x=a$, but is *not* "smooth" there, i.e. has a cusp(corner). Usually here either the two one-sided tangents exist at $x=a$ but have different slopes,

or there is a vertical tangent at $x = a$. In the former case, this will translate into a jump discontinuity of $f'(x)$; in the latter case, this translates into a vertical asymptote of $f'(x)$.

- $f(x)$ looks "smooth" at $x = a$, but has a vertical tangent there. Again, this will translate into a vertical asymptote of $f'(x)$.

(b) After marking all x's where $f'(x)$ does not exist (including possible vertical asymptotes, etc.), we move on to graphing $f'(x)$ where it exists. First find all places where the tangents to $f(x)$ are horizontal and mark the corresponding 0's on the graph of f'. Next determine the intervals where $f(x)$ increases i.e. has positive tangent slopes, and where $f(x)$ decreases, i.e. has negative tangent slopes. In the former case, $f'(x)$ will be positive, and in latter case, $f'(x)$ will be negative. In each such interval, answer the following two questions: whether the tangent slopes are positive or negative, and whether the tangent slopes themselves are increasing or decreasing. Translate this into the corresponding property of $f'(x)$.

(c) For more precise drawing, in each of the above intervals mark several tangent lines, guestimate their slopes and mark the corresponding points on the graph of $f'(x)$. Connect all these marked points to obtain the graph of $f'(x)$. Don't forget the places where $f'(x)$ was not defined!

(7) **How do we find derivatives using DLs?** If you are given $f(x)$ via one formula and you are not asked to use the definition of derivative, you apply DLs.

(a) First see if you can further simplify the given formula. In particular, try to avoid applying the Quotient Rule whenever possible because it is prone to errors. In practice this mean: try to get rid of denominators by either splitting fractions and then simplifying each fraction separately (see formulas for fraction manipulations below), or by direct cancellation of common stuff in the numerator and denominator, or by moving the denominator into the numerator: e.g. x^3 in the denominator becomes x^{-3} in the numerator.

(b) If you are going to apply the Power Rule, turn all expressions like $\sqrt[n]{x}^m$ into the standard form $x^{\frac{m}{n}}$. Again, such expressions in the denominator should move into the numerator wherever suitable by flipping the sign of the power: $\sqrt[n]{x}^m$ in the denominator becomes $x^{-\frac{m}{n}}$ in the numerator.

(c) Look at your function $f(x)$ to figure out its components, the simpler pieces it is made of, and decide which DL(s) you are going to use. In some cases, you may have to apply several DLs one after the other, so keep good track of your intermediate results, or else your calculations will be untraceable. A good strategy is to name some of the simpler components of $f(x)$, e.g. $g(x)$, $h(x)$, etc. and perform some of the necessary differentiation on these functions on the side and then put back your results together. To reduce errors and to make clear that you do know the DLs, it is always good to write the DL formula in terms of functions at first, e.g.

$$((5x+2)\cdot x^3)' \overset{PR}{=} (5x+2)'\cdot x^3 + (5x+2)\cdot (x^3)' = ...$$

(d) In case your function is given by several formulas on different intervals, you must find the derivative of each such formula on the corresponding interval. In the end, you must compare your results for the left-side and right-side derivative at the "break" points to determine if you function is differentiable there. E.g. if $f(x)$ is defined by two different formulas on $(2,5]\cup(5,8)$, then at the end you must compare $f'_-(5) \overset{?}{=} f'_+(5)$. If yes, then $f'(5)$ also exists; if not, then $f'(5)$ doesn't exist. You final answer for $f'(x)$ is again going to be given by several different formulas on the corresponding intervals.

(8) **How do we use implicit differentiation?** This is used to find tangent lines and their slopes to *curves* in the plane which are *not* graphs of functions (i.e. they violate the vertical line test). Thus, we don't have a function formula to differentiate, but instead an equation for the curve, e.g.

$x^3+x^2y+4y^2=6$. It will be hard, sometimes impossible, to solve such an equation for y, and hence a formula for y may not be available. [1]

(a) We imagine that y is given by such a formula $y=y(x)$ (e.g. $x^3+x^2y(x)+4y(x)^2=6$), and we differentiate (with respect to x) both sides of the given equation, e.g.

$$(x^3+x^2y(x)+4y^2(x))'=(6)' \Rightarrow 3x^2+2xy+x^2y'+8y\cdot y'=0$$

Do *not* forget to include y' wherever appropriate, due to the Chain Rule.

(b) Solve the above for y':

$$3x^2+2xy+y'(x^2+8y)=0 \Rightarrow y'=-\frac{3x^2+2xy}{x^2+8y}.$$

This is the best we can do for y': we have expressed it in terms of x and the original function y.

(c) If we are asked something about derivatives, slopes and tangents at specific places, then we use the above formula for y' and if necessary, the original equation for y. E.g. in our example, find the slope and the equation for the tangent at point $(1,1)$. [2] Now we use the formula for $y'(x)$ and substitute $x=1$, $y=1$:

$$y'(1)=-\frac{3+2}{1+8}=-\frac{5}{9}.$$

Finally, we use the point-slope formula:

$$y'(1)=\frac{y-y(1)}{x-1} \Rightarrow -\frac{5}{9}=\frac{y-1}{x-1} \Rightarrow y=-\frac{5}{9}x+\frac{14}{9}.$$

It is always good to check if this is the correct equation for the tangent line: yes, because the slope is $-5/9$, and if we plug in the point $(1,1)$ it works: $1=1$.

(d) Say, we want to find all points on the curve where the tangent to the curve is horizontal. In general, this is not an easy question to answer. Set $y'(x)=0$, and obtain two equations in terms of x and y: the derivative equation and the original equation. Now you are supposed to solve this system of two equations for x and y. In our example, this amounts to:

$$\left|\begin{array}{l} 0=-\frac{3x^2+2xy}{x^2+8y} \\ x^3+x^2y+4y^2=6 \end{array}\right.$$

The first equation yields $0=3x^2+2xy=x(3x+2y)$, i.e. $x=0$ or $y=-\frac{3}{2}x$. Substituting in the second equation: $4y^2=6$ (when $x=0$), or $x^3-\frac{3}{2}x^3+4\frac{9}{4}x^2=6$ (when $y=-\frac{3}{2}x$), i.e. $y=\pm\sqrt{\frac{3}{2}}$, while the second equation is a pain and I won't solve it here. The final answer would have been: the tangent lines to the curve are horizontal at points $(0,\sqrt{\frac{3}{2}})$, $(0,-\sqrt{\frac{3}{2}})$, and at the points yielded by the second case above.

The good news is that if a similar question appears on the exam, the calculations will be easier. The method, however, is outlined above. Note that similarly you can solve all sorts of questions about the tangents to such curves: e.g. find where the tangents are parallel to $y=x$ (set $y'(x)=1$), etc.

(9) **How do we find where $f(x)$ is increasing or decreasing?**

(a) Use the *Increasing/Decreasing Test.* If $f'(x)>0$ everywhere on some interval, then $f(x)$ increases on this interval. If $f'(x)<0$ everywhere on some interval, then $f(x)$ decreases on this interval. If $f'(x)=0$ everywhere on some interval, then $f(x)$ is constant on this interval. Thus, find for which x's $f'(x)>0$, for which $f'(x)<0$.

[1] In the particular example, you can indeed solve for y viewing the given equation as a quadratic equation in "y", but believe me, you probably don't want to do that, and you should follow the method of implicit differentiation instead.

[2] Note that this point was obtained by substituting $x=1$ into the original equation: $1+y+4y^2=6$ and obtaining solutions $y=1,-5/4$. Thus, you could have been asked to find instead the tangent line at point $(1,-5/4)$.

(b) Be careful when $f'(x_0) = 0$: just the fact that $f'(x_0) = 0$ does **not** mean that the function is constant nearby x! (E.g. x^3 nearby $x = 0$.) Thus, check the sign of the derivative at nearby points and this will tell you what is going on exactly at $f'(x_0)$. For example, if $f'(x) > 0$ both on the left and right of x_0, then the function is increasing at x_0 too. But if $f'(x)$ changes sign at x_0, then $f(x_0)$ is a local extremum so that $f(x)$ is neither increasing nor decreasing at x_0. □

(10) **How do we find the global extrema of $f(x)$?**

(a) First, find where $f'(x_0) = 0$ and record all such $f(x_0)$.

(b) Next, record all $f(x_0)$ where derivative does **not** exist, and also check and record what happens at the ends of the intervals (if there are any ends, that is.)

(c) Compare all values $f(x_0)$ at the critical points recorded above and find which are your global min/max. Be careful if the function is defined on an infinite interval or has vertical asymptotes. This gives opportunity for $f(x)$ to have arbitrarily large (small) values: some limit(s) of $f(x)$ could be $\pm\infty$, so in such cases, global min/max may not exist. □

(11) **How do we find the local extrema of $f(x)$?** For simplicity of exposition, let's assume that $f(x)$ is a nice function, so we don't have to worry for now about discontinuities or non-existing derivatives.

(a) First find the potential local extrema of $f(x)$, i.e. check where $f'(x_0) = 0$ and record all such places x_0.

(b) Next, use the *Second Derivative Test* (if $f''(x_0)$ exists, of course) to determine which of these potential extrema are realized local extrema. In particular, if $f''(x_0) > 0$ then $f(x_0)$ is a local minimum; if $f''(x_0) < 0$ then $f(x_0)$ is a local maximum.

(c) If $f''(x_0) = 0$ or the second derivative $f''(x_0)$ does not exists (the latter could happen even if $f(x)$ is a nice function!), then the Second Derivative Test fails to give us anything useful, so we try the *First Derivative Test*. If $f'(x)$ changes its sign at x_0 from $+$ to $-$, then $f(x_0)$ is a local maximum; if $f'(x)$ changes its sign at x_0 from $-$ to $+$, then $f(x_0)$ is a local minimum. If $f'(x)$ does **not** change its sign at x_0, then $f(x_0)$ is **not** a local extremum. □

(d) The *First Derivative Test* is **bulletproof!!!**

(12) **How do we find where the function is concave up or down?**

(a) Use the *Concavity Test*. If $f''(x) > 0$ everywhere on some interval, then $f(x)$ is concave-up on this interval. If $f''(x) < 0$ everywhere on some interval, then $f(x)$ is concave-down on this interval.

(b) If $f''(x_0) = 0$, then there could be an *inflection point* at x_0, so we check if $f''(x)$ changes sign at x_0. If yes, then at x_0 the function does indeed have an inflection point: the function changes from concave-up to concave-down or the other way around. But if $f''(x)$ does **not** change sign at x_0, then the function does **not** have an inflection point at x_0; and $f(x)$ continues to be concave-up (or concave-down) as it were before.

(c) As an example, compare the two functions x^3 and x^4 nearby $x_0 = 0$ and determine if they have inflection points at 0. Further, determine the intervals where these functions are concave-up or concave-down. □

4. Useful Formulas and Miscallaneous Facts

(1) **Quadratic formula**: useful for factoring quadratic polynomials as $a(x - x_1)(x - x_2)$, where x_1 and x_2 are the two roots of the polynomial, and a is the leading coefficient. Useful also for graphing quadratic polynomials: will yield the x-intercepts (or tell you that they don't exist.)

(2) **Rationalizing formula**: $\sqrt{A} - \sqrt{B} = \dfrac{(\sqrt{A} - \sqrt{B})(\sqrt{A} + \sqrt{B})}{\sqrt{A} + \sqrt{B}} = \dfrac{(A - B)}{\sqrt{A} + \sqrt{B}}$.

(3) **Factorization formulas**: $A^2 - B^2 = (A-B)(A+B)$ and $A^3 - B^3 = (A-B)(A^2+AB+B^2)$.

(4) **Binomial formulas:** $(A+B)^2 = A^2 + 2AB + B^2$, $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$.

(5) **Putting fractions under a common denominator**. The most general formula is as follows: $\frac{A}{B} + \frac{C}{D} = \frac{AD+BC}{BD}$. Yet, it is worth noting that if fractions already share something in their denominators, it will be faster to take this into account, e.g.

$$\frac{2x+1}{x^2} + \frac{x^3}{x(x-1)} = \frac{(2x+1)(x-1) + x\cdot x^3}{x^2(x-1)} = \frac{x^4+2x^2-x-1}{x^2(x-1)}.$$

(6) **Exponential Functions**: domains of definition, ranges, graphs; for which bases do these function increase/decrease.

(7) **Manipulations with Fractions**

(a) *Splitting fractions:* $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$; $\frac{ab}{cd} = \frac{a}{c}\cdot\frac{b}{d}$;

(b) **Wrong** *formula:* $\frac{a}{b+c} \neq \frac{a}{b} + \frac{a}{c}$.

(c) *Putting fractions under a common denominator:* $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.

(d) *When denominators have something in common:* $\frac{a}{be} + \frac{c}{de} = \frac{ad+bc}{bde}$.

(e) *"Fractions over fractions":* $\frac{a}{c} : \frac{b}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$; $\frac{a}{\frac{c}{d}} = \frac{ad}{c}$; $\frac{\frac{a}{b}}{c} = \frac{a}{bc}$.

(8) **Manipulations with Exponentials**

$$a^{b+c} = a^b \cdot a^c,\ \frac{a^b}{a^c} = a^{b-c},\ (a^b)^c = a^{bc},\ a^{\frac{b}{c}} = \sqrt[c]{a^b},\ \frac{1}{\sqrt[c]{a^b}} = \frac{1}{a^{\frac{b}{c}}} = a^{-\frac{b}{c}},\ a^0 = 1.$$

5. Exercises to Review

A good review of all homework exercises and examples from class should be an excellent preparation for the exam. It is important that you understand especially well the applications to business and economics, i.e. problems about cost, revenue, profit in different situations: airline companies, bridge tolls, stores, building fences, etc. etc.

6. Cheat Sheet

For the midterm, you are allowed to have a "cheat sheet" - *one page* of a regular 8×11 sheet. You can write whatever you wish there, under the following conditions:

- The whole cheat sheet must be **handwritten by your own hand**! No xeroxing, no copying, (and for that matter, no tearing pages from the textbook and pasting them onto your cheat sheet.)
- Any violation of these rules will disqualify your cheat sheet and may end in disqualifying your midterm. I may decide to randomly check your cheat sheets, so let's play it fair and square. :)
- Don't be a **freakasaurus**! Start studying for the exam several days in advance, and prepare your cheat sheet at least 2 days in advance. This will give you enough time to become familiar with your cheat sheet and be able to use it more efficiently on the exam.