Topics for Review for Midterm I in Calculus 16A

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Contents

1. Definitions

Be able to **write** precise definitions for any of the following concepts (where appropriate: both in words and in symbols), to **give** examples of each definition, to **prove** that these definitions are satisfied in specific examples. Wherever appropriate, be able to **graph** examples for each definition. What is/are:

(1) a *function*?
(2) an *independent* variable; a *dependent* variable?
(3) a *domain*; a *range* of a function?
(4) the *graph* of a function? the vertical line test?
(5) a *piecewise defined* function (p.21, Ex.4)?
(6) the function *absolute value* $|x|$?
(7) a *linear*, *quadratic*, *polynomial*, *rational* function? domains of these functions?
(8) a *power*, *exponential* function? domains of these functions?
(9) *zeros* and *graph* of a quadratic function?
(10) the *composition* $f \circ g$ of two functions $f(x)$ and $g(x)$? How to substitute for x into $f \circ g$?
(11) *interest rate* compounded annually, monthly, daily? *principal amount*? general formula for amount of money after a certain period of time? difference between *nominal* and *effective* interest rates?
(12) concepts and formulas for *circumference, areas, volumes* of various standard geometric objects?
(13) the *slope* of a line, the *equation* of a line and their geometric interpretations? what is $\Delta y/\Delta x$?
(14) the formula for the slope of a line given by two points on the line? the point–slope formula for a line?
(15) how to decide if a table of data corresponds to a linear function?
(16) interpretation of the y-intercept and the slope of a line in practical problems?
(17) the *tangent line* to the graph of a function $f(x)$ at $x = a$? a *secant line* of the graph of $f(x)$? what does it mean that $f(x)$ has a *vertical* tangent at $x = a$?
(18) the formula for *algebraic calculation* of derivative?
(19) the *derivative* $f'(a)$ at $x = a$? What does it mean that $f(x)$ is *differentiable* at a?
(20) the *derivative function* $f'(x)$? What does it mean that $f(x)$ is *differentiable* on interval (A, B)?
(21) the derivative of *basic* functions, e.g. constant, linear, power and polynomial function.
(22) the notation $\dfrac{d}{dx}$?
(23) *Limit* of $\lim\limits_{x\to\square_1} f(x) = \square_2$ where each of the "boxes" can be a finite number or $\pm\infty$?

Example. To say that $\lim\limits_{x\to -6} f(x) = 10$ means that:

(a) the y–values of $f(x)$ can be made as close to 10 as we please, provided x is close enough to -6.
(b) as x approaches ("horizontally") -6, the ("vertical" y–)values of $f(x)$ approach 10.

(24) *One-sided limit*, e.g. $\lim_{x\to a^-} f(x) = L$, $\lim_{x\to a^+} f(x) = -\infty$, etc.?

Example. To say that $\lim_{x\to 4^+} f(x) = 7$ means that

(a) $f(x)$ can be made as close to 7 as we please, provided x is close enough to 4 and $x > 4$.
(b) as x approaches ("horizontally") 4 from the right, the ("vertical" y–)values of $f(x)$ approach 7.

(25) *Vertical asymptote* of $f(x)$ at a? *Horizontal asymptote* of $f(x)$?

(26) *Continuity.* What does it mean that a function $f(x)$:

(a) is *continuous at* a?
(b) is *continuous on* (a,b), $[a,b)$, $[a,b]$, $(-\infty,b]$, (a,∞), etc.
(c) has a *removable discontinuity* at a? *jump discontinuity* at a? *infinite discontinuity* at a?

(27) the *total change* of $f(x)$ over $[a,b]$? the *average rate of change* of $f(x)$ over $[a,b]$? the *instantaneous rate of change* $f(x)$ at $x = a$?

(28) the *average velocity* and the *instantaneous velocity* of an object whose movement is given by $f(t)$? *acceleration*?

(29) the formula for *approximating the change of function* over $[a, a+h]$? How can we approximate the value $f(a+h)$ knowing $f(a)$ and $f'(a)$, when h is small? What is the formula and why does it work?

(30) the *marginal cost* and *marginal profit* in economics? How do we calculate them?

2. Theorems

Be able to **write** what each of the following theorems (laws, propositions, corollaries, etc.) says. Be sure to understand, distinguish and **state** the conditions (hypothesis) of each theorem and its conclusion. Be prepared to **give** examples for each theorem, and most importantly, to **apply** each theorem appropriately in problems. The latter means: decide which theorem to use, check (in writing!) that all conditions of your theorem are satisfied in the problem in question, and then state (in writing!) the conclusion of the theorem using the specifics of your problem.

(1) *(Finite) Limits Laws (LLs)*: addition, subtraction, multiplication, division, basic examples, multiplication by a constant, powers, roots. (Be careful about the division law! What extra conditions does it require?) $x \to \Box$ means: $x \to a$, $x \to \infty$, or $x \to -\infty$.

#	Theorem	Hypothesis	Conclusion
1	Linearity	$\lim_{x\to\Box} f_1(x) = L_1$, $\lim_{x\to\Box} f_2(x) = L_2$, $c_1, c_2 \in \mathbb{R}$	$\lim_{x\to\Box}(c_1 f_1(x) + c_2 f_2(x)) = c_1 L_1 + c_2 L_2$
2	Product	$\lim_{x\to\Box} f_1(x) = L_1$, $\lim_{x\to\Box} f_2(x) = L_2$	$\lim_{x\to\Box} f_1(x) f_2(x) = L_1 L_2$
3	Quotient	$\lim_{x\to\Box} f_1(x) = L_1$, $\lim_{x\to\Box} f_2(x) = L_2$, $f_2(x) \neq 0$ for $x \approx a$, $L_2 \neq 0$	$\lim_{x\to\Box} \frac{f_1(x)}{f_2(x)} = \frac{L_1}{L_2}$
4	∞-Limit Sum	$\lim_{x\to\Box} f_1(x) = \infty$, and $\lim_{x\to\Box} f_2(x) = \infty$ or $f_2(x)$ bounded below	$\lim_{x\to\Box} (f_1(x) + f_2(x)) = \infty$
5	∞-Limit Product	$\lim_{x\to\Box} f_1(x) = \infty$, and $\lim_{x\to\Box} f_2(x) = L > 0$ or $f_2(x)$ bounded below by $K > 0$	$\lim_{x\to\Box} f_1(x) f_2(x) = \infty$
6	∞-Reciprocal	$\lim_{x\to\Box} f(x) = \infty$	$\lim_{x\to\Box} \frac{1}{f(x)} = 0$
7	0-Reciprocal	$\lim_{x\to\Box} \frac{1}{f(x)} = 0$ and $f(x) > 0$	$\lim_{x\to\Box} f(x) = \infty$

(2) *Infinite Limit Laws (∞–LLs).* In the following infinite limit laws, an expression like "$(-\infty)+(-\infty) = -\infty$" does **not** have a meaning on its own, except *in context*, i.e. it refers only to the following situation and to nothing else:

Theorem "$(-\infty) + (-\infty) = -\infty$". "If for functions $f(x)$ and $g(x)$ we know that $\lim_{x\to\Box} f(x) = -\infty$, $\lim_{x\to\Box} g(x) = -\infty$, then $f(x)+g(x)$ also has a limit when $x \to \Box$: this limit is $\lim_{x\to\Box}(f(x) + g(x)) = -\infty$."

Note that there are no infinite limit laws for substraction, division, or multilication of the type $0 \cdot \infty$, i.e. the symbolic expressions $\infty - \infty, \infty/\infty, 0 \cdot \infty$ do not make sense, and they are called *indeterminate*.

Name	∞-LL: Formula	Example
1. addition	$\infty + \infty = \infty$	$\lim_{x\to\infty}(x+x^2) = \lim_{x\to\infty} x + \lim_{x\to\infty} x^2 = \infty + \infty \overset{\infty\text{LL}}{=} \infty$
	$(-\infty) + (-\infty) = -\infty$	$\lim_{x\to\infty}(-x-x^2) = \lim_{x\to\infty}(-x) + \lim_{x\to\infty}(-x^2) = (-\infty)+(-\infty) \overset{\infty\text{LL}}{=} -\infty$
	$\infty - \infty$ undefined	**Never use ∞-LLs in such cases.**
2. multiplication	$\infty \cdot \infty = \infty$	$\lim_{x\to\infty} x^2 = \lim_{x\to\infty} x \cdot \lim_{x\to\infty} x = \infty \cdot \infty \overset{\infty\text{LL}}{=} \infty$
	$(-\infty)\cdot(-\infty) = \infty$	$\lim_{x\to\infty}(-x)^2 = \lim_{x\to\infty}(-x) \cdot \lim_{x\to\infty}(-x) = (-\infty)\cdot(-\infty) \overset{\infty\text{LL}}{=} \infty$
	$\infty \cdot (-\infty) = -\infty$	$\lim_{x\to\infty} -x^2 = \lim_{x\to\infty} x \cdot \lim_{x\to\infty}(-x) = \infty \cdot (-\infty) \overset{\infty\text{LL}}{=} -\infty$
3. multiplication by constant	$c \cdot \infty = +\infty$ if $c > 0$	$\lim_{x\to\infty}(2x) = 2\lim_{x\to\infty} x = 2\cdot\infty \overset{\infty\text{LL}}{=} \infty$
	$c \cdot \infty = -\infty$ if $c < 0$	$\lim_{x\to\infty}(-2x) = -2\lim_{x\to\infty} x = -2\cdot\infty \overset{\infty\text{LL}}{=} -\infty$
	$0 \cdot \infty$ undefined	**Never use ∞-LLs in such cases.**
4. addition by constant	$c + \infty = +\infty\ \forall c$	$\lim_{x\to\infty}(-2+x) = -2 + \lim_{x\to\infty} x = -2+\infty \overset{\infty\text{LL}}{=} \infty$
	$c - \infty = -\infty\ \forall c$	$\lim_{x\to\infty}(-2-x) = -2 - \lim_{x\to\infty} x = -2-\infty \overset{\infty\text{LL}}{=} -\infty$
5. division by $\pm\infty$	$\frac{c}{+\infty} = 0\ \forall c$	$\lim_{x\to\infty}\frac{2}{x} = \frac{2}{\lim_{x\to\infty} x} = \frac{2}{\infty} \overset{\infty\text{LL}}{=} 0$
	$\frac{c}{-\infty} = 0\ \forall c$	$\lim_{x\to\infty}\frac{2}{-x} = \frac{2}{\lim_{x\to\infty}(-x)} = \frac{2}{\infty} \overset{\infty\text{LL}}{=} 0$
6. division by $0^{\pm}$	$\frac{c}{0^+} = +\infty\ \forall c > 0$	$\lim_{x\to 0}\frac{2}{x^2} = \frac{2}{0^+} \overset{\infty\text{LL}}{=} +\infty$, $\lim_{x\to 0}\frac{-2}{x^2} = \frac{-2}{0^+} \overset{\infty\text{LL}}{=} -\infty$
	$\frac{c}{0^-} = -\infty\ \forall c > 0$	$\lim_{x\to 0}\frac{2}{-x^2} = \frac{2}{0^-} \overset{\infty\text{LL}}{=} -\infty$, $\lim_{x\to 0}\frac{-2}{-x^2} = \frac{-2}{0^-} \overset{\infty\text{LL}}{=} +\infty$
	$\frac{0}{0}, \frac{c}{0}$ undefined $\forall c$	**Never use ∞-LLs in such cases.**
7. basic cases	$\lim_{x\to\infty}\frac{1}{x} = 0$, $\lim_{x\to-\infty}\frac{1}{x} = 0$	
	$\lim_{x\to\infty} x = \infty$, $\lim_{x\to-\infty} x = -\infty$	
	$\lim_{x\to 0^+}\frac{1}{x} = +\infty$, $\lim_{x\to 0^-}\frac{1}{x} = -\infty$, $\lim_{x\to 0}\frac{1}{x}$ **does not exist.**	

(3) *Basic Continuous Functions.* All of the following types of functions are *continuous* on their respective domains of definition:

#	Function	Algebraic Formula and Conclusion	Follows from
1	Constants	c continuous at $\forall x$	LL for constants
2	Linear	$ax+b$ continuous at $\forall x$	LL for linear fn's
3	Quadratic	ax^2+bx+c continuous at $\forall x$	LL for quadratic fn's
4	Power	x^n continuous at $\forall x$, $\forall n = 1,2,3,...$	LL for powers
5	Polynomial	$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ continuous at $\forall x$	LL for polynomials
6	Rational	$\frac{f(x)}{g(x)}$ continuous where $g(x) \neq 0$ ($f(x), g(x)$ - poly's)	LL for ratio, CL for poly's
7	Root	$\sqrt[n]{x}$ continuous at $\forall x$ where defined	LL for roots
8	Exponential	a^x continuous at $\forall x$ $(a > 0)$	LL for exponentials

(4) *"Double-sided" Theorems for Limits and Continuity.*
Example. The limit of $f(x)$ when $x \to a$ exists if and only if the two one-sided limits of $f(x)$ at a exist and are equal to each other. In this case, the limit of $f(x)$ exists and equals their common value. In symbols: $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x)$.

(5) *Theorem I.(Differentiable $\Rightarrow$ continuous.)* If $f(x)$ is differentiable at a, then $f(x)$ is continuous at a. If $f(x)$ is differentiable everywhere on its domain, then it is also continuous everywhere on its domain.

(6) *Contrapositive Theorem II. (Non-differentiable $\Rightarrow$ non-continuous.)* If $f(x)$ is *not* continuous at a, then $f(x)$ is *not* differentiable at a.

(7) *Converse Statement is False!* Continuity does not guarantee differentiability. Counterexample?

(8) *Differentiation Laws (DLs).*

   (a) *Basis Cases:* constant, quadratic and polynomial functions.

   (b) *Power Rule:* $(x^c)' = c\,x^{c-1}$ for any constant c.

   (c) *Multiplication by a Constant:* If $f(x)$ is a differentiable function, then $(c\,f(x))' = c\,f'(x)$.

   (d) *Sum and Difference Rules*: If $f(x)$ and $g(x)$ are differentiable, then their sum and difference are also differentiable: $(f(x)+g(x))' = f'(x)+g'(x)$, and $(f(x)-g(x))' = f'(x)-g'(x)$.

3. Problem Solving Techniques

(1) **How do we find $\lim\limits_{x\to a} f(x)$ when the (finite) LLs fail?**

   (a) If $f(x) = \dfrac{g(x)}{h(x)}$ and "plugging in a" yields $\dfrac{0}{0}$, try factoring polynomials and rationalizing expressions with square roots. The idea is to end up with $(x-a)$ both in numerator and denominator, cancel it, and then again attempt to apply LLs.

   (b) If $f(x)$ is a piecewise-defined function (i.e. given by different formulas on several intervals), try first to find the left-hand and the right-hand limits separately, and then compare them to see if they are equal (or if they exist, for that matter).

   (c) If $f(x)$ is given by a formula involving absolute values, again proceed by finding and comparing the two one-sided limits.

(2) **How do we determine if a function is continuous at a?** By definition of continuity, there are 3 things to check:

   (a) Find $\lim\limits_{x\to a} f(x)$ by following either limits laws or the techniques suggested above. (If it doesn't exist, then the function has no chance of being continuous at a. If it exists but is an infinite limit $\pm\infty$, again the function is not continuous at a; in fact, it has an *infinite discontinuity* at a.)

   (b) Find $f(a)$. If f is not defined at a, then the function is not continuous at a.

   (c) If the above two steps yield two finite numbers, compare them to check if they are equal: $\lim\limits_{x\to a} f(x) = f(a)$. If yes, the function is continuous at a; if not, the function is not continuous at a.

(3) **How do we find $\lim\limits_{x\to \square_1} f(x)$ when infinite limits are involved, but ∞–LLs fail?**

   (a) When finding the limit of a rational function: $\lim\limits_{x\to\pm\infty} \dfrac{P_1(x)}{P_2(x)}$ (here $P_1(x)$ and $P_2(x)$ are polynomials), we know that ∞/∞ doesn't make sense. So, we factor out the highest powers of x from both top and bottom polynomials, cancel, and then apply LLs again. (Note: In the end, all that will matter will be the leading terms of the two polynomials - no other terms will survive the above operations.) Similar ideas apply to any other fractions which involve polynomials and possibly radicals.

   (b) If ∞–LLs produce expressions involving $\infty - \infty$, we know that this doesn't make sense, so we look for a different approach. If polynomials are involved, factoring out the highest power is a good start. If square roots are involved, rationalizing might help. If two or more fractions are involved, putting them under a common denominator to arrive at one single fraction is the first step; then apply other techniques mentioned above.

   (c) If ∞–LLs produce an expression of the type $0\cdot\infty$, we know that this doesn't make sense, so we look for a different approach. Each example of this type has to be considered individually; most likely, we will end up factoring or rationalizing in search of common things to cancel, and after that we will attempt again to apply LLs.

(4) **How do we find derivatives from the definition?** Read carefully if you are being asked to find a specific derivative $f'(a)$, or the *whole* derivative function $f'(x)$. In each case, you have two choices how to proceed, as listed below.

(a) $f'(a) = \lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a}\cdot$ Here a is a constant and x moves towards a, so we expect that x will disappear and a will remain in the final result for $f'(a)$.

(b) $f'(a) = \lim\limits_{h\to 0} \dfrac{f(a + h) - f(a)}{h}\cdot$ Here a is a constant and h moves towards 0, so we expect that h will disappear and a will remain in the final result for $f'(a)$. This formula is nothing else but formula (a) where x is replaced by $a + h$.

(c) $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x + h) - f(x)}{h}\cdot$ Here x is viewed as a constant and h moves towards 0, so we expect that h will disappear and x will remain in the final result for the derivative function $f'(x)$. This formula is nothing else but formula (b) where a is replaced by x.

(d) One can also find $f'(x)$ by first finding $f'(a) = \lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a}$; in the result of this calculation x will disappear and only a will remain; in this final formula replace a by x to obtain a formula for the whole derivative function $f'(x)$.

(5) **How do we find equations of tangent lines?**

(a) First find the corresponding derivative $f'(a)$: this will be the slope of your tangent line.

(b) Next use the point–slope formula for the point $P(a, f(a))$ and for the slope $f'(a)$ from part (a):

$$f'(a) = \frac{y - f(a)}{x - a} \quad \Leftrightarrow \quad y - f(a) = f'(a)(x - a)\cdot \tag{1}$$

Here a and $f(a)$ are constants, and x and y are variables in the equation for your tangent line. Where suitable, multiply through and simplify to obtain a formula of the type:

$$y = f'(a) \cdot x + b.$$

(6) **How do we find all the tangents lines to the graph of $f(x)$ which are parallel to some line $y = mx + b$?**

(a) First find the slope m of the given line; be careful with this since the line may not be given in the standard "line-equation" form as above and you may have to rewrite it.

(b) Find the derivative $f'(x)$.

(c) Next, set $f'(x) = m$, where m is the found slope of the line above, and solve it for x.

(d) Finally, let's say you found several solutions x_1, x_2... etc. What remains is to find the corresponding points on the graph of $f(x)$ through which the wanted tangent lines will pass: $P_1(x_1, f(x_1))$, $P_2(x_2, f(x_2))$, etc. If the problem asks for finding the equation of these tangent lines, well, proceed now with the point-slope formula as before.

(7) **How do we find derivatives using DLs?** If you are given $f(x)$ via one formula and you are not asked to use the definition of derivative, you apply DLs. (You **cannot** apply DL's which we haven't covered yet in class, e.g. you **cannot** use DL's for product, quotient, chain rule, etc. If you use any of those, you will received 0 points on the corresponding problem! If the problem asks to use the definition of derivative, that's what you muts to use!)

(8) If you are going to apply the Power Rule, turn all expressions like $\sqrt[n]{x}^m$ into the standard form $x^{\frac{m}{n}}$. Again, such expressions in the denominator should move into the numerator wherever suitable by flipping the sign of the power: $\sqrt[n]{x}^m$ in the denominator becomes $x^{-\frac{m}{n}}$ in the numerator.

(9) In case your function is given by several formulas on different intervals, you must find the derivative of each such formula on the corresponding interval. In the end, you must compare your results for the left-side and right-side derivative at the "break" points to determine if you function is differentiable there. E.g. if $f(x)$ is defined by two different formulas on $(2, 5] \cup (5, 8)$, then at the end you must

compare $f'_-(5) \stackrel{?}{=} f'_+(5)$. If yes, then $f'(5)$ also exists; if not, then $f'(5)$ doesn't exist. You final answer for $f'(x)$ is again going to be given by several different formulas on the corresponding intervals.

4. Useful Formulas and Miscallaneous Facts

(1) **Quadratic formula**: useful for factoring quadratic polynomials as $a(x-x_1)(x-x_2)$, where x_1 and x_2 are the two roots of the polynomial, and a is the leading coefficient. Useful also for graphing quadratic polynomials: will yield the x-intercepts (or tell you that they don't exist.)

(2) **Rationalizing formula**: $\sqrt{A}-\sqrt{B} = \dfrac{(\sqrt{A}-\sqrt{B})(\sqrt{A}+\sqrt{B})}{\sqrt{A}+\sqrt{B}} = \dfrac{(A-B)}{\sqrt{A}+\sqrt{B}}$.

(3) **Factorization formulas**: $A^2-B^2=(A-B)(A+B)$ and $A^3-B^3=(A-B)(A^2+AB+B^2)$.

(4) **Binomial formulas:** $(A+B)^2 = A^2+2AB+B^2$, $(A+B)^3 = A^3+3A^2B+3AB^2+B^3$.

(5) **Putting fractions under a common denominator**. The most general formula is as follows: $\dfrac{A}{B}+\dfrac{C}{D}=\dfrac{AD+BC}{BD}$. Yet, it is worth noting that if fractions already share something in their denominators, it will be faster to take this into account, e.g.

$$\frac{2x+1}{x^2}+\frac{x^3}{x(x-1)} = \frac{(2x+1)(x-1)+x\cdot x^3}{x^2(x-1)} = \frac{x^4+2x^2-x-1}{x^2(x-1)}.$$

(6) **Manipulations with Fractions**

(a) *Splitting fractions:* $\dfrac{a+b}{c} = \dfrac{a}{c}+\dfrac{b}{c}$; $\dfrac{ab}{cd}=\dfrac{a}{c}\cdot\dfrac{b}{d}$;

(b) **Wrong** *formula:* $\dfrac{a}{b+c} \neq \dfrac{a}{b}+\dfrac{a}{c}$.

(c) *Putting fractions under a common denominator:* $\dfrac{a}{b}+\dfrac{c}{d}=\dfrac{ad+bc}{bd}$.

(d) *When denominators have something in common:* $\dfrac{a}{be}+\dfrac{c}{de}=\dfrac{ad+bc}{bde}$.

(e) *"Fractions over fractions":* $\dfrac{a}{c}:\dfrac{b}{d}=\dfrac{\frac{a}{b}}{\frac{c}{d}}=\dfrac{ad}{bc}$; $\dfrac{a}{\frac{c}{d}}=\dfrac{ad}{c}$; $\dfrac{\frac{a}{b}}{c}=\dfrac{a}{bc}$.

I cannot conceive of any other operation on fractions! If you think of one, let me know!

(7) **Manipulations with Exponentials**

$a^{b+c}=a^b\cdot a^c$, $\dfrac{a^b}{a^c}=a^{b-c}$, $(a^b)^c=a^{bc}$, $a^{\frac{b}{c}}=\sqrt[c]{a^b}$, $\frac{1}{\sqrt[c]{a^b}}=\frac{1}{a^{\frac{b}{c}}}=a^{-\frac{b}{c}}$, $a^0=1$.

5. Exercises to Review

Review **all** homework and quiz problems and solutions. These should be sufficient to do well on the exam.

6. Cheat Sheet

For the midterm, you are allowed to have a "cheat sheet" - *one page* of a regular 8×11 sheet. You can write whatever you wish there, under the following conditions:

- The whole cheat sheet must be **handwritten by your own hand**! No xeroxing, no copying, (and for that matter, no tearing pages from the textbook and pasting them onto your cheat sheet.)
- Any violation of these rules will disqualify your cheat sheet and may end in disqualifying your midterm. I may decide to randomly check your cheat sheets, so let's play it fair and square. :)
- Don't be a **freakasaurus**! Start studying for the exam several days in advance, and prepare your cheat sheet at least 2 days in advance. This will give you enough time to become familiar with your cheat sheet and be able to use it more efficiently on the exam.