HOW NOT TO PROVE THE POINCARÉ CONJECTURE

John R. Stallings

Introduction

I have committed — the sin of falsely proving Poincaré's Conjecture. But that was in another country; and besides, until now no one has known about it.

Now, in hope of deterring others from making similar mistakes, I shall describe my mistaken proof. Who knows but that somehow a small change, a new interpretation, and this line of proof may be rectified!

In the back of my mind when I conceived my proof was this theorem.

Theorem 0. *(For $n \neq 2$). Let $f : M \to K$ be a map of a connected orientable n-manifold into an n-complex, and let $C_1, \dots, C_k$ be some of the n-simplexes of K such that the degree of f on each C_i is zero (that is, the homology map induced by f, $H_n(M) \to H_n(K, K - \text{int } C_i)$, is zero. Suppose f induces a homomorphism of $\pi_1(M)$ onto $\pi_1(K)$. Then f is homotopic to a map into $K - (\text{int } C_1 \cup \dots \cup \text{int } C_k)$.*

A special argument establishes this theorem for $n = 1$. For $n \geq 3$ we may argue as follows. We shall make a number of changes on f which will be independent of each other, since $n \geq 3$; hence we need only consider the case that $k = 1$ and suppose that C_1 is covered twice with opposite orientations by $f(M)$. The inverse image of a small cell in C_1 is the union of two cells A and B in M. Let P be a path in M from A to B; fP represents an element of $\pi_1(K)$. Since $\pi_1(M) \to \pi_1(K)$ is onto, we can modify P by adding on a loop whose image represents the inverse of fP; thus we can suppose fP is a null-homotopic loop in K.

Since the dimension of M is at least 3, we can choose P to be a non-singular path and change f by a homotopy in the neighborhood of P so that $f(P) \subset C_1$. If T is a tube around P, then $A \cup T \cup B$ will be an n-cell mapped into C_1 with degree zero. A further homotopy within $A \cup T \cup B$ will uncover a point of C_1; by pushing away from that point, we uncover all of int C_1.

But, in my proof of Poincaré's Conjecture, I need this theorem for $n = 2$. The argument above fails in this case for several reasons. We cannot uncover 2-cells independently of each other; we cannot make the path P non-singular; if P were non-singular, the homotopy bringing $f(P)$ into C_1 might cause us to cover up cells which we want to uncover.

The reader may be able to patch up some of these points. If he patches up all these points, he will have proved the Poincaré Conjecture (for we shall show how

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Theorem 0 for $n = 2$ implies the Poincaré Conjecture) incorrectly. For, Theorem 0 is false for $n = 2$: Consider a torus with two 2-cells C_1 and C_2 attached to kill the fundamental group; there is a map of the 2-sphere into this complex; by a homotopy we can uncover either C_1 or C_2, but not both simultaneously.

1. A conjecture about the 3-sphere

In the 3-sphere S^3 let T be a tame 2-manifold such that both components of $S^3 - T$ have free fundamental groups. Let U and V denote the closures of the components of $S^3 - T$.

According to theorems of Papakyriakopoulos, both U and V are handlebodies. The only conceivable such embedding of T is shown in Fig. 1.

[Figure 1 is a picture of a standard 2-manifold embedded in R^3, bounding a handlebody, with a curve on it separating it into two pieces, such that in each half of the complement in R^3, the curve bounds a disk.]

However, if the genus of T is greater than 1, this standard embedding has a property which does not seem to follow immediately from the fact that U and V are handlebodies. Namely, there is a simple closed curve C on T, not contractible on T, yet bounding 2-cells both in U and in V.

Conjecture A. *The existence of such a curve C can be proved only from the hypothesis that both U and V are handlebodies.*

If we hope that Conjecture A is true, a reasonable direction to attempt a proof of the Poincaré Conjecture can be made as follows.

Poincaré's Conjecture is that any simply-connected 3-manifold M is a 3-sphere. It is known that any orientable 3-manifold such as M, has a Heegaard representation as $U \cup V$, where U and V are handlebodies and $U \cap V$ is their common boundary, a 2-manifold T.

If the genus of T should happen to be one, then M is a lens space, and so, if simply-connected, is a 3-sphere.

Assume that we could prove Conjecture A for M, rather than for the 3-sphere: That is, if the genus of T is greater than one, then on T there is a simple closed curve C, not contractible on T, yet bounding 2-cells in both U and V. Then we could write M as the connected sum $M_1 \# M_2$ of two manifolds whose Heegaard representations would have less genus. And so by induction on the genus, we would know that M is indeed a 3-sphere.

2. Reduction to group theory

Let $M = U \cup V$, $T = U \cap V$ be a Heegaard representation of a 3-manifold. We obtain a diagram of fundamental groups, with homomorphisms induced from inclusions:

$$\begin{array}{ccccc} \pi_1(U) & \xleftarrow{\varphi} & \pi_1(T) & \xrightarrow{\psi} & \pi_1(V) \\ & \searrow & \downarrow & \swarrow & \\ & & \pi_1(M) & & \end{array}$$

Since φ and ψ are homomorphisms onto, it follows from van Kampen's Theorem that $\pi_1(M)$ is isomorphic to the quotient of $\pi_1(T)$ by $(\ker\,\varphi)\cdot(\ker\,\psi)$. Hence M is simply connected exactly when

$$\pi_1(T) = (\ker\,\varphi)\cdot(\ker\,\psi).$$

We have an obvious homomorphism to investigate,

$$\varphi\times\psi : \pi_1(T) \to \pi_1(U)\times\pi_1(V).$$

The kernel of this homomorphism is clearly,

$$\ker\,\varphi\times\psi = (\ker\,\varphi)\cap(\ker\,\psi).$$

Therefore our geometric problem has been "reduced", by virtue of Dehn's Lemma, to the more algebraic problem: Does $\ker\,\varphi\times\psi$ contain an element which can be represented by a simple closed curve on T?

Now this cannot be true for arbitrary 3-manifolds, for if it were we would have proved, "Every 3-manifold is a connected sum of lens spaces", which is absurd.

Theorem 1. *$\varphi\times\psi$ is a homomorphism onto, if and only if M is simply connected.*

First, if $\varphi\times\psi$ is onto, since $\pi_1(U)\times\pi_1(V)$ is the product of the kernels of the projections onto its factors, it follows that $\pi_1(T)$ is the product of $\ker\,\varphi$ and $\ker\,\psi$, and hence M is simply connected.

Conversely, if M is simply connected, then $\pi_1(T) = (\ker\,\varphi)\cdot(\ker\,\psi)$. Let (α,β) be an arbitrary element of $\pi_1(U)\times\pi_1(V)$. Since φ and ψ are onto, there are α_1, β_1 in $\pi_1(T)$ such that $\varphi(\alpha_1)=\alpha$ and $\psi(\beta_1)=\beta$. We can decompose α_1 and β_1 thus: $\alpha_1 = x\alpha_2$, $\beta_1=\beta_2 y$, where x and β_2 belong to $\ker\,\varphi$, and α_2 and y belong to $\ker\,\psi$. Then $\varphi\times\psi(\alpha_2\cdot\beta_2) = (\alpha,\beta)$. Hence $\varphi\times\psi$ is onto.

Now our "reduction" can be stated as the following conjecture:

Conjecture B. *Let T be an orientable 2-manifold of genus $n>1$. Let F_1 and F_2 be free groups of rank n. Let $\eta : \pi_1(T)\to F_1\times F_2$ be a homomorphism onto. Then there is a non-trivial element of* $\ker\,\eta$ *which is represented by a simple closed curve on T.*

We have shown that Conjecture B implies both the Poincaré Conjecture and Conjecture A. It is likely that from the data of Conjecture B we can reconstruct a 3-manifold; in which case, then, conversely, the Poincaré Conjecture and Conjecture A together would imply Conjecture B.

3. Finding simple closed curves.

Papakyriakopoulos and Maskit have discovered an interesting characterization of simple closed curves in terms of planar covering spaces. Their results show that Conjecture B is equivalent to the following:

Conjecture C. *In the situation of Conjecture B, there is a non-trivial normal subgroup N of $\pi_1(T)$, such that $N \subset \ker \eta$ and such that the covering space of T which corresponds to N is a planar surface.*

However, in our discussion we are interested in a different characterization of simple closed curves.

Let us say that a homomorphism $\varphi : G \to A*B$ of a group G into a free product of groups is *essential* if there is no element $x \in A*B$ such that $x \cdot \varphi(G) \cdot x^{-1}$ is contained in one of the factors A or B.

Theorem 2. *If $G = \pi_1(T)$, where T is a closed 2-manifold, and if $\varphi : G \to A*B$ is an essential homomorphism, then there is some non-trivial element of* $\ker \varphi$ *which is represented by a simple closed curve on T.*

Proof. : Represent $A*B$ as the fundamental group of $X = X_A \cup X_B$ where X_A and X_B are open sets in X with fundamental groups A and B respectively, and where $X_A \cap X_B$ is simply connected. Represent φ by a continuous function $f : T \to X$.

We can divide T into submanifolds T_A and T_B, whose intersection is their common boundary, such that $f(T_A) \subset X_A$ and $f(T_B) \subset X_B$. The components of $T_A \cap T_B$ are simple closed curves, whose images by f lie in $X_A \cap X_B$ and hence are contractible in X.

If, contrary to the conclusion we wish to draw, every such simple closed curve were trivial on T, then any one, say C, would bound a 2-cell D. Redefine f on D so as to map D into $X_A \cap X_B$, and then redefine T_A and T_B; this will reduce the number of components of $T_A \cap T_B$. Finally, we obtain a map f' and a division of T so that either T_A or T_B is empty. f' induces the same homomorphism on fundamental groups as f does, modulo an inner automorphism (we may have moved the basepoint around), so that φ is inessential, contradicting hypothesis.

Reinterpreting Conjecture B in the light of this theorem, we have:

Conjecture D. *In the situation of Conjecture B, the map $\eta : \pi_1(T) \to F_1 \times F_2$ can be factored through an essential map of $\pi_1(T)$ into some free product $A*B$.*

Thus have we replaced the purely geometric Poincaré Conjecture by the purely algebraic Conjecture D.

4. Geometric "proof" of Conjecture D

Let us consider the map $\eta : \pi_1(T) \to F_1 \times F_2$ with components φ and ψ, so that $\eta(x) = (\varphi(x), \psi(x))$. It is possible given that η is onto, after a moderate amount of algebraic slickness, to find a presentation of $\pi_1(T)$ as

$$\{a_1, b_1, \ldots, a_n, b_n : \prod_{i=1}^{n} [a_i, b_i] = 1\}$$

and bases $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\beta_1, \ldots, \beta_n\}$ of F_1 and F_2 such that, *modulo the commutator subgroups*, $\eta(a_i) \equiv (\alpha_i, 1)$ and $\eta(b_i) \equiv (1, \beta_i)$.

Now interpret F_1 as the fundamental group of a bouquet of circles $X_1 \vee X_2 \vee \cdots \vee X_n$, where X_i corresponds to the basis element α_i. Similarly interpret F_2 as the fundamental group of $Y_1 \vee \cdots \vee Y_n$, with Y_i corresponding to β_i. Define $W = (X_1 \vee \cdots \vee X_n) \times (Y_1 \vee \cdots \vee Y_n)$.

We may interpret η as the homomorphism induced from a function $f : T \to W$. Because of our clever choice of bases, f will have degree zero on the tori $X_i \times Y_j$ for $i \neq j$ and degree one on the tori $X_i \times Y_i$.

Let W^* denote $(X_1 \times Y_1) \cup (X_2 \times Y_2) \cup \cdots \cup (X_n \times Y_n)$. This is the union of n tori with a single point in common.

If only, alas, Theorem 0 were valid for dimension two, we could conclude that the map $f : T \to W$ could, up to homotopy, be factored through a map $g : T \to W^*$. The map g would have to induce an isomorphism on 1-dimensional homology, and hence, writing $\pi_1(W^*)$ as the obvious free product, the map of fundamental groups would be essential.

And so Conjecture D would be proved. Poincaré's Conjecture would follow. Fame and Fortune would be ours.

5. Conclusion

There are two points about this incorrect proof worthy of note.

The first is that when we try to prove Theorem 0 in dimension two, we always run up against the problem of trying to simplify, by some geometric trick, the situation. But any little homotopy that would simplify the picture always in fact, greatly complicates it. This phenomenon has characterized every attempt that I have made or heard of to prove Poincaré's Conjecture. This is the place to look for flaws in any asserted "proof".

The second point is that I was unable to find flaws in my "proof" for quite a while, even though the error is very obvious. It was a psychological problem, a blindness, an excitement, an inhibition of reasoning by an underlying fear of being wrong. Techniques leading to the abandonment of such inhibitions should be cultivated by every honest mathematician.

6. Afterword

This is a TeXed version (November 2000) of the original [3], which was written when the author was at Princeton University.

References

[1] B. Maskit, *A theorem on planar covering surfaces with applications to 3-manifolds*, Ann. of Math. **81** (1965), 341–355.

[2] C. D. Papakyriakopoulos, *A reduction of the Poincaré Conjecture to group-theoretic conjectures*, Ann. of Math. **77** (1963), 250–305.

[3] John R. Stallings, *How not to prove the Poincaré Conjecture*, Topology Seminar Wisconsin, 1965, Ann. of Math. Studies, vol. 60, 1966, pp. 83–88.

Mathematics Dept., University of California, Berkeley, CA 94720

E-mail address: stall@math.berkeley.edu