A finite difference approach to the infinity Laplace equation and tug-of-war games

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References

1. A., S. A finite difference approach to the infinity Laplace equation and tug-of-war games, preprint, arXiv:0906.2871
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Main ideas

1. Changing the rules of the original tug-of-war game (as introduced by Peres, Schramm, Sheffield, and Wilson) in an ε-thick strip near the boundary, we obtain a new tug-of-war game with value functions which are continuous. This continuity simplifies many arguments, and in particular we can prove that the new tug-of-war game has a value in more general situations than for the original game.
2. By the dynamic programming principle, the value functions satisfy a certain finite difference equation. Studying the finite difference equation directly yields easier proofs of old results, some new results, and a useful new perspective.

Main ideas

3. By slightly modifying subsolutions of the (continuum) infinity Laplace equation, we obtain subsolutions of the finite difference equation. This allows us to reduce some problems for the continuum equation to those for the finite difference equation, which is easier to analyze.

Preliminaries

We are interested in the Dirichlet problem for the normalized infinity Laplace equation, which is,

$$\begin{cases} -\Delta_\infty u = f & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega, \end{cases}$$

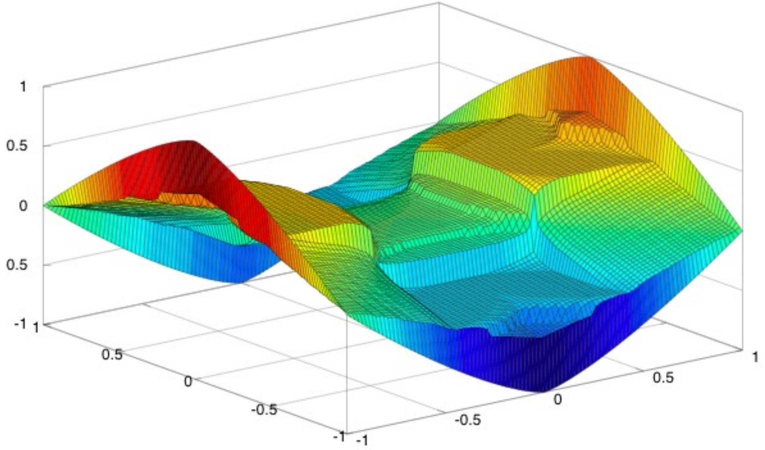
where $-\Delta_\infty u = -|Du|^{-2} \sum u_{x_i x_j} u_{x_i} u_{x_j}$, $f \in C(\Omega) \cap L^\infty(\Omega)$, and $g \in C(\partial\Omega)$.

For simplicity, in this talk we assume the domain $\Omega \subseteq \mathbb{R}^n$ is convex.

We also assume everyone is familiar with the original ε-step tug-of-war game introduced by Peres, Schramm, Sheffield, and Wilson, whose value functions approximate solutions the above Dirichlet problem for f satifying $f > 0$, $f < 0$, or $f \equiv 0$.

Discontinuous value functions

Standard tug-of-war with $\Omega = (-1,1)^2$, $g(x,y) = x^{4/3} - y^{4/3}$, $f \equiv 0$, and $\varepsilon = 3/5$.



Boundary-biased tug-of-war

The value functions for standard tug-of-war are generally discontinuous at the boundary $\partial\Omega$, and these discontinuities propagate inward.

We propose the following modified tug-of-war game, which we call *boundary-biased tug-of-war*.

Before a coin is tossed, and the token is at $x \in \Omega$, the players choose points $x_I, x_{II} \in \bar{\Omega}(x,\varepsilon) = \bar{B}(x,\varepsilon) \cap \bar{\Omega}$. Now a biased coin is tossed, and the token is moved to x_I with probability P, and to x_{II} with probability $1 - P$, where

$$P = \frac{\rho_\varepsilon(x, x_{II})}{\rho_\varepsilon(x, x_I) + \rho_\varepsilon(x, x_{II})},$$

and

$$\rho_\varepsilon(x,y) := \begin{cases} \max(\varepsilon, |x-y|) & \text{if} \quad x, y \in \Omega, \\ |x-y| & \text{if} \quad x \in \partial\Omega \quad \text{or} \quad y \in \partial\Omega. \end{cases}$$

Boundary-biased tug-of-war

We also change the accumulation of running payoff. If the game terminates during stage $N-1$ with $x_N \in \partial\Omega$, player II pays player I the quantity

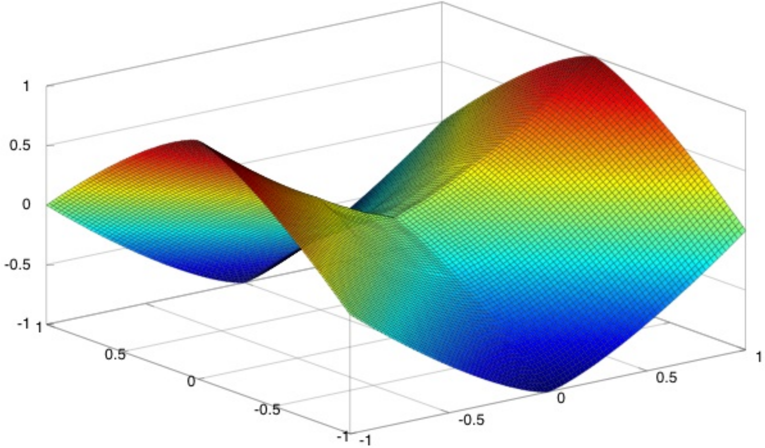
$$g(x_N) + \sum_{k=1}^{N} \frac{\varepsilon \rho_\varepsilon(x_{k-1}, x_k)}{2} f(x_{k-1}).$$

Observe that the rules are identical to the original tug-of-war game while the token lies in $\Omega_\varepsilon = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \geq \varepsilon\}$. We have only changed the rules in the strip $\Omega \setminus \Omega_\varepsilon$ so as to give a player wishing to exit the domain (and end the game) a better probability of winning the coin toss. The effect of this change is to "pull" the value function to the boundary conditions.

The idea to change the rules near the boundary was inspired by Oberman's numerical scheme.

Boundary-biased tug-of-war

Boundary-biased tug-of-war with $\Omega = (-1, 1)^2$,
$g(x, y) = x^{4/3} - y^{4/3}$, $f \equiv 0$, and $\varepsilon = 3/5$.



Boundary-biased tug-of-war

In some sense, our modification is the best we could hope for.

Theorem

The value functions for boundary-biased ε-step tug-of-war are continuous. (Even in the case that the value functions for Player I and Player II are not equal.)

Theorem

If $f \equiv 0$ and the boundary data g is Lipschitz, then the value function is minimizing Lipschitz.

The finite difference infinity Laplace equation

By the dynamic programming principle, the value functions for boundary-biased ε-step tug-of-war satisfy

$$\begin{cases} -\Delta_\infty^\varepsilon u = f & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega, \end{cases}$$

where

$$\Delta_\infty^\varepsilon u(x) := \frac{1}{\varepsilon}\left(S_\varepsilon^+ u(x) - S_\varepsilon^- u(x)\right),$$

$$S_\varepsilon^+ u(x) := \sup_{y \in \bar{\Omega}(x,\varepsilon)} \frac{u(y) - u(x)}{\rho_\varepsilon(x,y)},$$

and

$$S_\varepsilon^- u(x) := \sup_{y \in \bar{\Omega}(x,\varepsilon)} \frac{u(x) - u(y)}{\rho_\varepsilon(x,y)},$$

We call $\Delta_\infty^\varepsilon$ the *finite difference infinity Laplacian*.

Existence of solutions

Using PDE-style arguments, and in particular the Perron method, we obtain the following theorem.

Theorem

For any $g \in C(\partial\Omega)$, $f \in C(\Omega) \cap L^\infty(\Omega)$, and $\varepsilon > 0$, there exists a unique maximal solution $\bar{u}$ and a unique minimal solution $\underline{u}$ of

$$\begin{cases} -\Delta_\infty^\varepsilon u = f & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega. \end{cases}$$

The maximal and minimal solutions are continuous, and in fact the maximal solution $\bar{u}$ is the value function for Player II, and the minimal solution $\underline{u}$ is the value function for Player I.

A comparison principle for the finite difference equation

Definition

Suppose $u \in C(\bar{\Omega})$. A strict ε-local maximum of u in Ω is a non-empty closed set $F \subseteq \Omega$ such that $u(y) < \max_F u$ whenever $y \in \bar{\Omega} \setminus F$ and $\operatorname{dist}(y, F) \leq \varepsilon$.

Theorem

Suppose $u, v \in C(\bar{\Omega})$ and

$$-\Delta_\infty^\varepsilon u \leq -\Delta_\infty^\varepsilon v \quad \textit{in} \quad \Omega.$$

If u has no strict ε-local maximum in Ω, or v has no strict ε-local minimum in Ω, then

$$\max_{\bar{\Omega}}(u - v) = \max_{\partial\Omega}(u - v).$$

Proof of the comparison principle

We generalize an argument due to Le Gruyer, who proved a comparison principle on finite graphs.

Suppose that $\max_{\bar{\Omega}}(u - v) > \max_{\partial\Omega}(u - v)$.

Define $E := \{x \in \bar{\Omega} : (u - v)(x) = \max_{\bar{\Omega}}(u - v)\}$, and $F := \{x \in E : u(x) = \max_E u\}$. Note that E and F are closed and non-empty subsets of Ω.

Claim 1: $S_\varepsilon^- u = S_\varepsilon^- v$ and $S_\varepsilon^+ u = S_\varepsilon^+ v$ on E.

If $x \in E$, then $u - v$ attains its maximum at x. This implies $S_\varepsilon^- u(x) \geq S_\varepsilon^- v(x)$ and $S_\varepsilon^+ u(x) \leq S_\varepsilon^+ v(x)$. Thus $-\Delta_\infty^\varepsilon u(x) \geq -\Delta_\infty^\varepsilon v(x)$ for any $x \in E$. By hypothesis, $-\Delta_\infty^\varepsilon u(x) \leq -\Delta_\infty^\varepsilon v(x)$, so these inequalities must be equalities.

Proof of the comparison principle

Claim 2: F is a strict ε-local maximum of u.

Otherwise, we may choose $y \in \bar{\Omega} \setminus F$ such that $u(y) \geq \max_F u$, and $y \in \bar{\Omega}(x, \varepsilon)$ for some $x \in F$. It follows that $y \notin E$.
We may assume that $S^+_\varepsilon u(x) = \rho_\varepsilon(x, y)^{-1}(u(y) - u(x))$.
Since $y \notin E$, we must have $u(y) - v(y) < u(x) - v(x)$.
Thus $v(y) - v(x) > u(y) - u(x)$ and so $S^+_\varepsilon v(x) > S^+_\varepsilon u(x)$, contradicting Claim 1. $\square$

Game interpretation of comparison

Suppose $\bar{u} \neq \underline{u}$ in boundary-biased tug-of-war.

Since $\max_{\bar{\Omega}}(\bar{u} - \underline{u}) > 0 = \max_{\partial\Omega}(\bar{u} - \underline{u})$, the comparison theorem implies that $\underline{u}$ has a strict ε-local minimum in Ω and $\bar{u}$ has a strict ε-local maximum in Ω.

Thus, there are closed sets on which each player would like to keep the token, so as to accumulate running payoff. The player who is required to guarantee termination of the game is forced to give up opportunity of accumulating running payoff in order to move the token to the boundary.

This should only happen when there is a sign-changing running payoff f, whose values are large relative to the oscillation of the boundary condition g.

Comparison for nonnegative running payoff

The following easy lemma allows us to derive a more familiar comparison result from the general one above.

Lemma

If $u \in C(\bar{\Omega})$ and $-\Delta_\infty^\varepsilon u \le 0$ in Ω, then u has no strict ε-local maximum in Ω. Similarly, if $v \in C(\bar{\Omega})$ and $-\Delta_\infty^\varepsilon v \ge 0$ in Ω, then v has no strict ε-local minimum in Ω.

Corollary

If $f \in C(\Omega) \cap L^\infty(\Omega)$ does not change sign in Ω and $u, v \in C(\bar{\Omega})$ satisfy

$$-\Delta_\infty^\varepsilon u \le f \le -\Delta_\infty^\varepsilon v \quad \text{in} \quad \Omega,$$

then $\max_{\bar{\Omega}}(u - v) = \max_{\partial\Omega}(u - v)$.

Uniqueness for small running payoff

Using a compactness argument, we can prove the following uniqueness result.

Theorem

Suppose $g \in C(\partial\Omega)$ is not constant on any neighborhood of $\partial\Omega$. Then there exists a $\delta = \delta(\varepsilon, g) > 0$ such that

$$\begin{cases} -\Delta_\infty^\varepsilon u = f & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega, \end{cases}$$

has a unique solution, provided that $\|f\|_{L^\infty(\Omega)} < \delta$.

Question

Can $\delta > 0$ be chosen to depend only on $(\max g - \min g)$? Can we prove a similar statement for the continuum equation?

Continuity estimates

By studying the analogue of gradient flow lines for our finite difference equation, we can prove the following estimate.

Theorem

Suppose $g \in C(\partial\Omega)$, $f \in C(\Omega) \cap L^\infty(\Omega)$, $\varepsilon > 0$ *and* $u : \bar{\Omega} \to \mathbb{R}$ *is bounded and satisfies*

$$\begin{cases} -\Delta_\infty^\varepsilon u = f & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega. \end{cases}$$

There is a function $\omega \in C[0,\infty)$ *depending only on* $\|f\|_{L^\infty(\Omega)}$, g, *and* Ω *such that* $\omega(0) = 0$ *and*

$$|u(x) - u(y)| \leq \omega(\rho_\varepsilon(x,y)) \quad \text{for every} \quad x, y \in \bar{\Omega}.$$

Continuity estimates

Observe that we only have uniform estimates in the ρ_ε metric, even though the value functions are known to be continuous.

As previously stated, when $f \equiv 0$ and g is Lipschitz, we know u is minimizing Lipschitz.

This leads to the following natural question.

Question

Are there continuity estimates for the finite difference infinity Laplace equation which are uniform in ε when $f \not\equiv 0$?

Favored games

Before moving on to the continuum equation, we point out an interesting idea, which appears to be a kind of analytic analogue of the "favored games" introduced by Peres, Schramm, Sheffield, and Wilson.

Proposition

Suppose $u \in C(\Omega)$ *satisfies*

$$-\Delta_\infty^\varepsilon u \le f \quad \text{in} \quad \Omega_\varepsilon.$$

Define $u^\varepsilon(x) := \max_{\bar{B}(x,\varepsilon)} u$ *for each* $x \in \Omega_\varepsilon$*, and* $f^{2\varepsilon}(x) := \max_{\bar{B}(x,2\varepsilon)} f$ *for each* $x \in \Omega_{2\varepsilon}$*. Then*

$$-\Delta_\infty^{2\varepsilon} u^\varepsilon \le f^{2\varepsilon} \quad \text{in} \quad \Omega_{3\varepsilon}.$$

The continuum analogue of this idea turns out to be quite useful.

Convergence of the finite difference equation

The proposition on the previous slide can be used to prove the following convergence result.

Theorem

If $\varepsilon_k \downarrow 0$, $u_k \in C(\Omega)$, $-\Delta_\infty^{\varepsilon_k} u_k \le f$ in Ω, and $u_k \to u \in C(\Omega)$ locally uniformly, then u satisfies

$$-\Delta_\infty u \le f \quad in \quad \Omega.$$

Corollary

There exists a solution of

$$\begin{cases} -\Delta_\infty u = f & in \quad \Omega, \\ u = g & on \quad \partial\Omega, \end{cases}$$

for any $g \in C(\partial\Omega)$ and $f \in C(\Omega) \cap L^\infty(\Omega)$.

Maxing over ε-balls

Definition

For functions $u, v \in C(\Omega)$, we define $u^{\varepsilon}(x) := \max_{\bar{B}(x,\varepsilon)} u$ and $v_{\varepsilon}(x) := \min_{\bar{B}(x,\varepsilon)} v$ for every $x \in \Omega_{\varepsilon}$. To avoid conflict with the notation for sequences, we only use ε for this.

Theorem

If $u \in C(\Omega)$ satisfies

$$-\Delta_{\infty} u \leq f \quad \text{in} \quad \Omega,$$

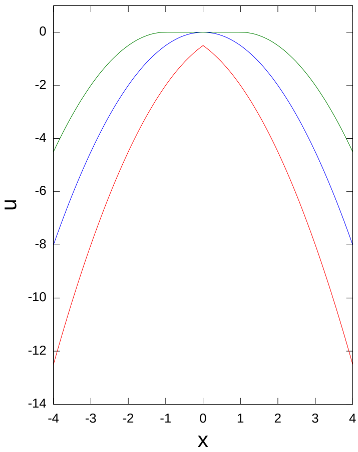
then

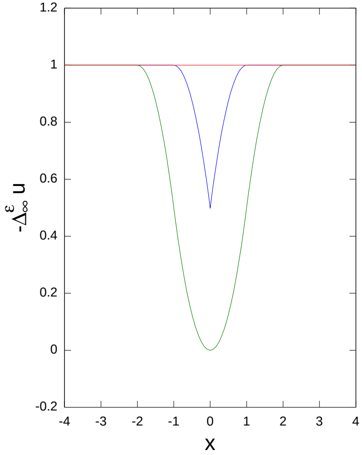
$$-\Delta_{\infty}^{\varepsilon} u^{\varepsilon} \leq f^{2\varepsilon} \quad \text{in} \quad \Omega_{2\varepsilon}.$$

We may obtain the corresponding result for supersolutions v by applying the above theorem to $-v$.

Maxing over ε-balls

The function $u(x) := -\frac{1}{2}|x|^2$ satisfies $-\Delta_\infty u = 1$ in $\mathbb{R}^n$. When $\varepsilon = 1$, the functions u^ε and u_ε satisfy $-\Delta_\infty^\varepsilon u^\varepsilon \leq 1$ and $-\Delta_\infty^\varepsilon u_\varepsilon = 1$ in $\mathbb{R}^n$.





Jensen's theorem

An immediate application of the ε-ball theorem is a new proof of Jensen's theorem on the uniqueness of infinity harmonic functions.

Theorem (Jensen)

If $u, v \in C(\bar{\Omega})$ *and*

$$-\Delta_\infty u \leq 0 \leq -\Delta_\infty v \quad \textit{in} \quad \Omega,$$

then

$$\max_{\bar{\Omega}}(u - v) = \max_{\partial\Omega}(u - v).$$

PDE proofs appeared in Jensen (1993), Barles and Busca (2001), Aronsson, Crandall, and Juutinen (2004), Crandall, Gunnarsson, and Wang (2007). A probability proof appeared in Peres, Schramm, Sheffield, and Wilson (2009).

Jensen's theorem

Proof.

Suppose $u, v \in C(\bar{\Omega})$ satisfy

$$-\Delta_\infty u \leq 0 \leq -\Delta_\infty v \quad \text{in} \quad \Omega.$$

The ε-ball theorem implies

$$-\Delta_\infty^\varepsilon u^\varepsilon \leq 0 \leq -\Delta_\infty^\varepsilon v_\varepsilon \quad \text{in} \quad \Omega_{2\varepsilon}.$$

Comparison for the finite difference equation then implies

$$\sup_{\Omega_\varepsilon}(u^\varepsilon - v_\varepsilon) = \sup_{\Omega_\varepsilon \setminus \Omega_{2\varepsilon}} (u^\varepsilon - v_\varepsilon).$$

Sending $\varepsilon \to 0$ yields

$$\max_{\bar{\Omega}}(u - v) = \max_{\partial\Omega}(u - v). \quad \square$$

Convergence of the finite difference equation

Next, we use the ε-ball theorem and a perturbed test function argument in the spirit of Evans (1989) to prove the following.

Theorem

If $\varepsilon_k \downarrow 0$, $u_k \in C(\Omega)$, $-\Delta_\infty^{\varepsilon_k} u_k \leq f$ in Ω, and $u_k \to u \in C(\Omega)$ locally uniformly, then u satisfies

$$-\Delta_\infty u \leq f \quad in \quad \Omega.$$

We must check: If $\varphi \in C^\infty(\Omega)$ and $u - \varphi$ has a strict local maximum at some $x_0 \in \Omega$, then

$$-\Delta_\infty^+ \varphi(x_0) \leq f(x_0),$$

where

$$\Delta_\infty^+ \varphi(x_0) = |D\varphi(x_0)|^{-2} \langle D^2\varphi(x_0) D\varphi(x_0), D\varphi(x_0) \rangle$$

when $D\varphi(x_0) \neq 0$ and otherwise

$$\Delta_\infty^+ \varphi(x_0) = \max\{\langle D^2\varphi(x_0)v, v\rangle : v \in \partial B(0,1)\}.$$

Convergence of the finite difference equation

Proof.

Suppose $\varphi \in C^\infty(\Omega)$ and $u - \varphi$ has a strict local maximum at $x_0 \in \Omega$.
Since φ is smooth, it satisfies

$$-\Delta_\infty \varphi \geq -\Delta_\infty^- \varphi \geq -\Delta_\infty^+ \varphi \quad \text{in} \quad \Omega.$$

Applying the ε-ball theorem, we obtain

$$-\Delta_\infty^{\varepsilon_k} \varphi_{\varepsilon_k} \geq \left(-\Delta_\infty^+ \varphi\right)_{2\varepsilon_k} \quad \text{in} \quad \Omega_{2\varepsilon_k}.$$

We may assume $u_k - \varphi_{\varepsilon_k}$ attains its maximum in $\bar{\Omega}(x_k, \varepsilon_k)$ at $x_k \in \Omega$ and $x_k \to x_0$.
Thus,

$$\left(-\Delta_\infty^+ \varphi\right)_{2\varepsilon_k}(x_k) \leq -\Delta_\infty^{\varepsilon_k} \varphi_{\varepsilon_k}(x_k) \leq -\Delta_\infty^{\varepsilon_k} u_k(x_k) \leq f(x_k).$$

Sending $k \to \infty$ yields $-\Delta_\infty^+ \varphi(x_0) \leq f(x_0)$. □

Continuous dependence and uniqueness

Definition

We say that *ε-uniform continuous dependence holds at $f \in C(\Omega)$ and $g \in C(\partial\Omega)$* if there is a function $H \in C[0,\infty)$ such that $H(0) = 0$ and

$$\|u - \tilde{u}\|_{L^\infty(\Omega)} \le H\left(\varepsilon + \|f + \Delta_\infty^\varepsilon u\|_{L^\infty(\Omega)} + \|f + \Delta_\infty^\varepsilon \tilde{u}\|_{L^\infty(\Omega)}\right)$$

for every $\varepsilon > 0$ and $u, \tilde{u} \in C(\bar{\Omega})$ where $u = \tilde{u} = g$ on $\partial\Omega$.

Theorem

If ε-uniform continuous dependence holds at $f \in C(\bar{\Omega})$ and $g \in C(\partial\Omega)$, then

$$\begin{cases} -\Delta_\infty u = f & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega, \end{cases}$$

has a unique solution.

Continuous dependence and uniqueness

Using our theory for the finite difference equation, it is easy to show the following.

Proposition

Suppose $f \in C(\Omega) \cap L^\infty(\Omega)$ *and* $g \in C(\partial\Omega)$. *If* $f > a$ *or* $f < -a$ *some constant* $a > 0$, *then continuous dependence holds at* f *and* g *via*

$$H(s) := C(\|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\partial\Omega)})a^{-1}s,$$

where $C > 0$ *depends only on* $\mathrm{diam}(\Omega)$.

Continuous dependence and uniqueness

Together with the previous result, this constitutes a new proof of the following.

Theorem (Peres, Schramm, Sheffield, and Wilson)

If $g \in C(\partial\Omega)$, $f \in C(\Omega) \cap L^\infty(\Omega)$, and $f > 0$ or $f < 0$, then

$$\begin{cases} -\Delta_\infty u = f & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega. \end{cases}$$

has a unique solution.

Lu and Wang (2008) found another proof of this result using viscosity solution methods.

Other applications

We make use of the ε-ball theorem and our results for the finite difference equation to prove several new results.

- (Stability) If $f_k \in C(\Omega) \cap L^\infty(\Omega)$, $u_k \in C(\Omega)$ satisfy

$$-\Delta_\infty u_k = f_k \quad \text{in} \quad \Omega,$$

  and $u_k \to u \in C(\Omega)$ and $f_k \to f \in C(\Omega) \cap L^\infty(\Omega)$ locally uniformly, then u satisfies $-\Delta_\infty u = f$ in Ω. Lu and Wang (2008) proved this in the case $f \equiv 0$, $f > 0$ or $f < 0$.
- (Existence) The Dirichlet problem

$$\begin{cases} -\Delta_\infty u = f & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega. \end{cases}$$

  has unique maximal and minimal solutions for any $f \in C(\Omega) \cap L^\infty(\Omega)$ and $g \in C(\partial\Omega)$.

Other applications

- (Generic uniqueness) For fixed $f \in C(\Omega) \cap L^\infty(\Omega)$ and $g \in C(\partial\Omega)$, the Dirichlet problem

$$\begin{cases} -\Delta_\infty u = f + c & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega, \end{cases}$$

  has a unique solution for all but at most countable many $c \in \mathbb{R}$. This answers a question posed by Peres, Schramm, Sheffield, and Wilson.

- (Continuous dependence from 0) For $\gamma \in \mathbb{R}$, let $u_\gamma \in C(\bar{\Omega})$ be the unique solution to

$$\begin{cases} -\Delta_\infty u_\gamma = \gamma & \text{in} \quad \Omega, \\ u_\gamma = g & \text{on} \quad \partial\Omega. \end{cases}$$

  There is a constant $C > 0$ depending only on $\operatorname{diam}(\Omega)$ such that

$$\|u_\gamma - u_0\|_{L^\infty(\bar{\Omega})} \leq C|\gamma|^{\frac{1}{3}} \left(1 + \|g\|_{L^\infty(\partial\Omega)}\right).$$

Proof of the ε-ball theorem when $c = 0$

Our proof uses the following now standard result.

Proposition

Suppose $u \in C(\bar{\Omega})$ satisfies

$$-\Delta_\infty u \leq 0 \quad \textit{in} \quad \Omega,$$

and

$$\varphi(x) := a + b|x - x_0|,$$

for some $a, b \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n \setminus \Omega$. Then

$$\max_{\bar{\Omega}}(u - \varphi) = \max_{\partial\Omega}(u - \varphi).$$

This fact is due to Jensen. The well-known converse, which we do not use, is due to Crandall, Evans, and Gariepy.

Proof of the ε-ball theorem when $c = 0$

Suppose $u \in C(\Omega)$, $-\Delta_\infty u \leq 0$ in Ω, and $x \in \Omega_{2\varepsilon}$. We compute,

$$\begin{aligned} -\varepsilon^2 \Delta_\infty u^\varepsilon(x) &= \varepsilon S_\varepsilon^- u^\varepsilon(x) - \varepsilon S_\varepsilon^+ u^\varepsilon(x) \\ &= (u^\varepsilon(x) - (u^\varepsilon)_\varepsilon(x)) - ((u^\varepsilon)^\varepsilon(x) - u^\varepsilon(x)) \\ &\leq 2u^\varepsilon(x) - u^{2\varepsilon}(x) - u(x) \\ &= 2u(y) - u(z) - u(x), \end{aligned}$$

where $y \in \bar{B}(x, \varepsilon)$ and $z \in \bar{B}(x, 2\varepsilon)$ are such that $u(y) = u^\varepsilon(x)$ and $u(z) = u^{2\varepsilon}(x)$. We may assume $u(z) > u(x)$. Consider the cone

$$\varphi(w) := u(x) + \frac{u(z) - u(x)}{2\varepsilon}|w - x|.$$

Since $\varphi \geq u$ on $\partial(B(x, 2\varepsilon) \setminus \{x\})$, $\varphi \geq u$ on $\bar{B}(x, 2\varepsilon)$.
Since $u(z) > u(x)$, it follows that $|z - x| = 2\varepsilon$, $u(z) = \varphi(z)$, and $u(y) \leq \varphi(y) \leq \varphi(\frac{1}{2}(x + z))$. Thus,

$$2u(y) - u(z) - u(x) \leq 2\varphi\left(\tfrac{1}{2}(x + z)\right) - \varphi(z) - \varphi(x) = 0. \qquad \square$$

Biased tug-of-war

In joint work with Stephanie Somersille (in preparation) we extend these techniques to biased tug-of-war. For example, the corresponding ε-ball theorem is the following.

Theorem

If $\beta \in \mathbb{R}$, $f \in C(\Omega)$, *and* $u \in C(\Omega)$ *satisfies*

$$-\Delta_\infty u - \beta|Du| \leq f \quad in \quad \Omega,$$

then

$$\frac{\beta}{e^{\beta\varepsilon} - 1} S_\varepsilon^- u^\varepsilon - \frac{\beta}{1 - e^{-\beta\varepsilon}} S_\varepsilon^+ u^\varepsilon \leq f^{2\varepsilon} \quad in \quad \Omega_{2\varepsilon}.$$

Open questions

Question

Does continuous dependence hold for $-\Delta_\infty^\varepsilon$ (hence uniqueness for the continuum equation) at any $f \in C(\Omega) \cap L^\infty(\Omega)$ which doesn't change sign?

Question

Suppose the continuum equation has two distinct solutions. Must one have a strict local maximum?

Some related work

- Peres and Sheffield (2008 and Scott's tutorial) used games with noise to study the p-Laplacian.
- Manfredi, Parviainen, and Rossi (one of Tuesday's talks) have also used games with noise to study the p-Laplacian.
- Peres, Pete, and Somersille (the previous talk) have used tug-of-war games with a biased coin to study the PDE $-\Delta_\infty u - \beta|Du| = 0$.
- Charro, García Azorero, and Rossi (2009) have obtained existence of solutions to mixed Dirichlet-Neumann problems for the infinity Laplace equation using tug-of-war games.