# Random Walks on a Square

Consider the rectangular grid $\{X_{ij} : 0 \leq i, j \leq n\}$ of points in $[0,1]^2$ given by

$$X_{ij} = (i/n, j/n).$$

A *random walk* on this grid is a path determined by the following random process. Say you are currently sitting at $X_{ij}$. If you are on the boundary, i.e. if $i$ or $j$ equals 0 or $N$, then stop. Otherwise jump to one of the neighbors $X_{i+1,j}, X_{i-1,j}, X_{i,j+1}, X_{i,j-1}$ at random, each with probability 1/4. Repeat.

Suppose $f : \partial[0,1]^2 \to \mathbb{R}$ is any function, where $\partial[0,1]^2$ is the boundary of the square. If you start a random walk at an interior point $X_{ij}$, what is the expected value of $f(X_{i^*j^*})$ where $X_{i^*j^*}$ is the endpoint of the walk? Experiment with several interesting $f$.

What happens in the limit as $N \to \infty$? How is this related to the partial differential equation

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0?$$

Experiment further by changing the shape of the domain and the transition probabilities.