Solutions to

HW 7

p. 217 #13 Section 5.2

(a) From $$I - I_n \approx \frac{c}{n^p} \qquad (5.39)$$

derive $$\frac{I - I_n}{I - I_{2n}} \approx 2^p.$$

Solution: With $2n$ instead of n we get

$$I - I_{2n} \approx \frac{c}{(2n)^p} \qquad (5.40)$$

and thus

$$\frac{I - I_n}{I - I_{2n}} \approx \frac{\frac{c}{n^p}}{\frac{c}{(2n)^p}} = 2^p.$$

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part (b)

Derive the computable estimate

$$\frac{I_{2n} - I_n}{I_{4n} - I_{2n}} \approx 2^p \quad , \text{ and thus}$$

$$p \approx \log\left(\frac{I_{2n} - I_n}{I_{4n} - I_{2n}}\right) \Big/ \log 2 .$$

<u>Solution:</u>

$$\frac{I_{2n} - I_n}{I_{4n} - I_{2n}} = \frac{(I - I_n) - (I - I_{2n})}{(I - I_{2n}) - (I - I_{4n})}$$

$$\approx \frac{\frac{c}{n^p} - \frac{c}{2^p n^p}}{\frac{c}{2^p n^p} - \frac{c}{4^p n^p}} = \frac{1 - \frac{1}{2^p}}{\frac{1}{2^p} - \frac{1}{4^p}}$$

$$= \frac{4^p - 2^p}{2^p - 1} = \frac{2^p(2^p - 1)}{2^p - 1} = 2^p .$$

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Suppose that for $i = 0, 1, 2, 3, \ldots, n$

$\hat{f}_i$ is approximately $f(x_i)$ in such a manner that

$$\left| f(x_i) - \hat{f}_i \right| \le \epsilon.$$

Let $I_n \stackrel{\text{def}}{=} h \sum_{j=0}^{n} w_j f(x_j)$

be an integration formula where the weights w_j are positive and such that the formula integrates degree zero polynomials exactly.

For instance I_n might be trapezoidal rule with $w_0 = \frac{1}{2}$, $w_1 = w_2 = \ldots = w_{n-1} = 1$, $w_n = \frac{1}{2}$, or

I_n might be Simpson's rule with

$w_n = w_0 = \frac{1}{3}$, $w_1 = w_3 = w_5 = \ldots = w_{n-1} = \frac{4}{3}$,

$w_2 = w_4 = \ldots = w_{n-2} = \frac{2}{3}$.

Both integration formulas have positive weights and both have degree of precision greater than or equal to zero (Simpson's rule has degree 3, trapezoidal has degree 1) and thus they both integrate the constant polynomials exactly.

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Let $\widehat{I}_n \stackrel{\text{def}}{=} h \sum_{j=0}^{n} w_j \widehat{f}_j$, the integration formula used with the approximate values $\widehat{f}_j$.

Then $$|I_n - \widehat{I}_n| = \left| h \sum_{j=0}^{n} w_j f(x_j) - h \sum_{j=0}^{n} w_j \widehat{f}_j \right|$$

$$= h \left| \sum_{j=0}^{n} w_j \left(f(x_j) - \widehat{f}_j \right) \right|$$

$$\overset{\substack{\text{Triangle} \\ \text{inequality}}}{\leq} h \sum_{j=0}^{n} \left| w_j \left(f(x_j) - \widehat{f}_j \right) \right|$$

$$= h \sum_{j=0}^{n} |w_j| \cdot \left| f(x_j) - \widehat{f}_j \right|$$

$$\leq h \sum_{j=0}^{n} |w_j| \, \varepsilon = \varepsilon h \sum_{j=0}^{n} |w_j|$$

$$\overset{w_j > 0}{=\!=} \varepsilon h \sum_{j=0}^{n} w_j = \varepsilon \left(h \sum_{j=0}^{n} w_j \cdot 1 \right)$$

$$\overset{\substack{\text{formula integrates } 1 \\ \text{exactly}}}{=\!=\!=} \varepsilon \int_a^b 1 \, dx = \varepsilon (b-a).$$

p. 241 #6 Section 5.4

(a) Consider $D_h f(x) \stackrel{\text{def}}{=} \dfrac{f(x+h) - f(x)}{h}$ (5.76).

We know by (5.77) that
$$f'(x) - D_h f(x) = -\frac{h}{2} f''(c) \text{ for some } c$$
between x and $x+h$, so the error in this method is approximately the first power of h.

Now consider $2D_h f(x) - D_{2h} f(x) =$
$$= 2\,\frac{f(x+h) - f(x)}{h} - \frac{f(x+2h) - f(x)}{2h} =$$
$$= \frac{4f(x+h) - f(x+2h) - 3f(x)}{2h}$$

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{f'''(\xi)}{6}h^3$$
for some ξ between x and $x+h$

and
$$f(x+2h) = f(x) + 2f'(x)h + 2f''(x)h^2 + \frac{8}{6}f'''(\eta)h^3$$
for some η between x and $x+2h$, so
$$2D_h f(x) - D_{2h} f(x) = \frac{4f(x+h) - f(x+2h) - 3f(x)}{2h}$$
$$= f'(x) + \left[\frac{1}{3}f'''(\xi) - \frac{2}{3}f'''(\eta)\right]h^2,$$
i.e. the error of $2D_h f(x) - D_{2h} f(x)$ is proportional to h^2, so it converges more rapidly to zero.

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part (b) Consider $\mathcal{D}_h f(x) \stackrel{\text{def}}{=} \dfrac{f(x+h) - f(x-h)}{2h}$.

We know that $f'(x) - \mathcal{D}_h f(x) = \dfrac{-h^2}{6} f'''(c)$

for some c between $x-h$ and $x+h$ by (5.85), so the error of $\mathcal{D}_h f(x)$ is proportional to the second power of h.

Consider the Richardson extrapolation formula

$$\frac{4}{3}\mathcal{D}_h f(x) - \frac{1}{3}\mathcal{D}_{2h} f(x) =$$

$$= \frac{4}{3} \cdot \frac{f(x+h) - f(x-h)}{2h} - \frac{1}{3} \cdot \frac{f(x+2h) - f(x-2h)}{4h}$$

$$= \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h}.$$

Let us write out Taylor expansions for $f(x+2h)$, $f(x+h)$, $f(x-h)$, and $f(x-2h)$ centered about x:

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(b) continued

$$f(x+2h) = f(x) + 2hf'(x) + 2h^2 f''(x) + \frac{4}{3}h^3 f'''(x) + \frac{2}{3}h^4 f^{(4)}(x) + \frac{4}{15}h^5 f^{(5)}(x) + \frac{4}{45}h^6 f^{(6)}(x) + \frac{8}{315}h^7 f^{(7)}(x) + \cdots$$

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + \frac{1}{6}h^3 f'''(x) + \frac{1}{24}h^4 f^{(4)}(x) + \frac{1}{120}h^5 f^{(5)}(x) + \frac{1}{720}h^6 f^{(6)}(x) + \frac{1}{5040}h^7 f^{(7)}(x) + \cdots$$

$$f(x-h) = f(x) - hf'(x) + \frac{1}{2}h^2 f''(x) - \frac{1}{6}h^3 f'''(x) + \frac{1}{24}h^4 f^{(4)}(x) - \frac{1}{120}h^5 f^{(5)}(x) + \frac{1}{720}h^6 f^{(6)}(x) - \frac{1}{5040}h^7 f^{(7)}(x) + \cdots$$

$$f(x-2h) = f(x) - 2hf'(x) + 2h^2 f''(x) - \frac{4}{3}h^3 f'''(x) + \frac{2}{3}h^4 f^{(4)}(x) - \frac{4}{15}h^5 f^{(5)}(x) + \frac{4}{45}h^6 f^{(6)}(x) - \frac{8}{315}h^7 f^{(7)}(x) + \cdots$$

and thus

$$\frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h} =$$

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$$\frac{-f(x+2h)+8f(x+h)-8f(x-h)+f(x-2h)}{12h}$$

$$=\frac{12h\,f'(x)-\frac{2}{5}h^5 f^{(5)}(x)+\cdots}{12h}$$

$$= f'(x)-\frac{1}{30}h^4 f^{(5)}(x)+\cdots$$

so we see that the error of this method is proportional to h^4, which is better than h^2.

p. 241 #6 Section 5.4 part (c)

Define $D_h^{(2)}f(x) = \dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2}$.

The Richardson extrapolation

$$\frac{4}{3} D_h^{(2)} f(x) - \frac{1}{3} D_{2h}^{(2)} f(x) =$$

$$\frac{4}{3} \; \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - \frac{1}{3} \; \frac{f(x+2h) - 2f(x) + f(x-2h)}{4h^2}$$

$$= \cancel{\frac{-f(x+2h) + 4f(x+h) - 6f(x) + 4f(x-h) - f(x-2h)}{3h^2}}$$

$$= \frac{-f(x+2h) + 16f(x+h) - 30f(x) + 16f(x-h) - f(x-2h)}{12h^2}$$

$$= \frac{12h^2 f''(x) - \frac{2}{15} h^6 f^{(6)}(x) + \cdots}{12h^2}$$

$$= f''(x) - \frac{1}{90} h^4 f^{(6)}(x) + \cdots$$

so the error is proportional to h^4, which is better than h^2.

p. 242, #10, Section 5.4

(a) $$f(x+2h) = f(x) + 2hf'(x) + 2h^2 f''(x) + \frac{4}{3}h^3 f'''(x) + \frac{2}{3}h^4 f^{(4)}(x) + \cdots$$

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + \frac{1}{6}h^3 f'''(x) + \frac{1}{24}h^4 f^{(4)}(x) + \cdots$$

so to get $$\frac{Af(x+2h) + Bf(x+h) + Cf(x)}{h^2} \approx f''(x)$$

we need $A+B+C=0$, $2A+B=0$, and
$2A + \frac{1}{2}B = 1$.

This has solution $A=1$, $B=-2$, $C=1$, i.e.

$$\frac{f(x+2h) - 2f(x+h) + f(x)}{h^2} = f''(x) + hf'''(x) + \cdots$$

so this is ~~equal to~~ approximately $f''(x)$ with an error which is proportional to h to the first power.

Not a very good method.

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part (b)

$$\frac{A\,f(x+3h) + B\,f(x+2h) + C f(x+h) + D\,f(x)}{h^2}$$

is approximately $f''(x)$ when, considering the Taylor series on the last page and

$$f(x+3h) = f(x) + 3h\,f'(x) + \frac{9}{2}h^2 f''(x) + \frac{9}{2}h^3 f'''(x) + \frac{27}{8}h^4 f^{(4)}(x) + \cdots$$

$$A + B + C + D = 0$$

$$3A + 2B + C = 0$$

$$\frac{9}{2}A + 2B + \frac{1}{2}C = 1$$

$$\frac{9}{2}A + \frac{4}{3}B + \frac{1}{6}C = 0$$

which has solution $A = -1$, $B = 4$ $C = -5$ $D = 2$

So $$\frac{-f(x+3h)+4f(x+2h)-5f(x+h)+2f(x)}{h^2}$$

$$= \frac{h^2 f''(x) - \frac{11}{12} h^4 f^{(4)}(x) + \cdots}{h^2}$$

$$= f''(x) - \frac{11}{12} h^2 f^{(4)}(x) + \cdots$$

so the error is proportional to h^2, which is pretty bad for such a complicated method.