Math 128A, Spring 2007

Homework 6 Solution

5.1.12 It's enough to check exactness for 1, x, x^2, x^3, etc. The degree of precision is the first n for which our rule is not exact for x^{n+1}.

$$\int_0^1 1dx = 1 \qquad \frac{1}{4}1 + \frac{3}{4}1 = 1$$
$$\int_0^1 xdx = 1/2 \qquad \frac{1}{4}0 + \frac{3}{4}2/3 = 1/2$$
$$\int_0^1 x^2dx = 1/3 \qquad \frac{1}{4}0 + \frac{3}{4}4/9 = 1/3$$
$$\int_0^1 x^3dx = 1/4 \qquad \frac{1}{4}1 + \frac{3}{4}8/27 = 2/9$$

so the degree is 2.

5.1.14 We have

$$\begin{aligned} I(f) &= \int_{-1}^1 f(x) \\ &\approx \int_{-1}^1 [f(-1/3)(3/2)(1/3-x) + f(1/3)(3/2)(x+1/3)]dx \\ &= f(-1/3) + f(1/3) \end{aligned}$$

and we can check the degree is 2 by the method of problem 12.

5.1.15 When $f(x) = 1$, we have $\int_{-1}^1 f(x)dx = 2 = 1 + 1 = f(-\beta) + f(\beta)$ for any β. When $f(x) = x$, we have $\int_{-1}^1 f(x)dx = 0 = -\beta + \beta = f(-\beta) + f(\beta)$ for any β. Hence the degree is at least 1.

To make it work for $f(x) = x^2$, we need $\int_{-1}^1 f(x)dx = 2/3 = f(-\beta) + f(\beta) = 2\beta^2$, i.e. $\beta = \pm 1/\sqrt{3}$. This choice of β is also exact for $f(x) = x^3$ but not for $f(x) = x^4$. Hence the degree is 3.

5.2.1b By (5.23) with $f(x) = e^{-x^2}$ and $[a,b] = [0,1]$,

$$I(f) - T_n(f) = -\frac{h^2(b-a)}{12}f''(c_n) = \frac{(4c_n^2-2)e^{-c_n^2}}{12n^2}$$

with $c_n \in [0,1]$. Since

$$|(4c_n^2-2)e^{-c_n^2}| \le 2$$

we get a bound

$$|I(f) - T_n(f)| \le \frac{1}{6n^2}$$

5.2.4 For the integral

$$I^{(2)} = \int_0^4 \frac{dx}{1+x^2}$$

the error formula (5.32) says

$$I^{(2)} - T_n(f) \approx -\frac{h^2}{12}[f'(b) - f'(a)] = \frac{32}{867n^2}$$

Tabulating the error and error estimates,

n	actual	estimate
2	-1.33e-1	9.23e-3
4	-3.59e-3	2.31e-3
8	5.64e-4	5.77e-4
16	1.44e-4	1.44e-4
32	3.60e-5	3.60e-5
64	9.01e-6	9.01e-6
128	2.25e-6	2.25e-6

5.2.5c As in the previous problem

$$I(f) - T_n(f) \approx -\frac{h^2}{12}[f'(b) - f'(a)] = -\frac{4e^{-4}}{3n^2}$$

We want $|I(f) - T_n(f)| \le 10^{-10}$, so we solve

$$\frac{4e^{-4}}{3n^2} \le 10^{-10}$$

and obtain

$$n \ge \sqrt{\frac{4e^{-4}}{3 \cdot 10^{-10}}} \approx 1.56 \times 10^4$$

5.2.7 (a) By (5.23)

$$I(f) - T_n(f) = -\frac{h^2(b-a)}{12} f''(c_n) = \frac{8}{12n^2}\frac{1}{c_n^2}$$

with $c_n \in [1,3]$. Since $1/c_n^2 \le 1$, we have

$$|I(f) - T_n(f)| \le \frac{2}{3n^2}$$

By (5.32)

$$I(f) - T_n(f) \approx \frac{h^2}{12}[f'(b) - f'(a)] = \frac{h^2}{18}$$

To insure that $|I(f) - T_n(f)| \le 5 \times 10^{-8}$ it is enough to find n so that

$$\frac{2}{3n^2} \le 5 \times 10^{-8}$$

or

$$n \geq \sqrt{\frac{2}{3 \cdot 5 \times 10^{-8}}} \approx 3.65 \times 10^3$$

(b) Same as part (a) but a little longer...

midpoint Interpolating f at c, c with $c = (a+b)/2$, we have

$$f(x) = p_1(x) + f[c, c, x](x-c)^2$$

Now,

$$\begin{aligned}
\int_a^b f(x)dx - \int_a^b p_1(x)dx &= \int_a^b f[c, c, x](x-c)^2 dx \\
&= f[c, c, \tilde{\xi}] \int_a^b (x-c)^2 dx \\
&= \frac{f''(\xi)}{2} \int_a^b (x-c)^2 dx \\
&= \frac{f''(\xi)}{2} \frac{h^3}{12} \\
&= \frac{h^3}{24} f''(\xi)
\end{aligned}$$

where $\int_a^b p_1(x)dx = M_1(f)$. This is the simple error formula.
For the composite,

$$I(f) - T_n(f) = \sum_{j=1}^{n} \frac{h^3}{24} f''(\xi_j) = n \frac{h^3}{24} f''(\xi) = \frac{h^2(b-a)}{24} f''(\xi)$$

For the asymptotic

$$\sum_{j=1}^{n} \frac{h^3}{24} f''(\xi_j) \approx \int_a^b \frac{h^2}{24} f'(x)dx = \frac{h^2}{24}(f'(b) - f'(a))$$