Kicked experiment

The cubic polynomial

$$p(x) = \frac{1}{1000}\left(\frac{1}{6}x^3 + \left(75\tfrac{5}{6}\right)x + 15\right)$$

fits the first 5 data points exactly in that

$p(0) = .015$
$p(1) = .091$
$p(2) = .168$
$p(3) = .247$
$p(4) = .329$,

and this is more impressive considering that a cubic cannot always be made to pass through 5 data points.

So this seems like a good fit.

$p(5) = .415$, so we say $.418$ is wrong, it should have been $.415$.

page 31 Exercise 9

$$1 + x^3 * \left(\frac{-1}{3!} + x^3 * \left(\frac{1}{6!} + x^3 * \left(\frac{-1}{9!} + x^3 * \left(\frac{1}{12!} + x^3 * \frac{-1}{15!} \right) \right) \right) \right)$$

with $x^3 = x * x * x$

Thus 7 multiplications

p. 169 # 7 (4.5.7)

$$\int_{-1}^{1} \frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}}\,dx =$$

$$= \int_{-1}^{1} \frac{\cos(n \arccos x)\,\cos(m \arccos x)}{\sqrt{1-x^2}}\,dx$$

$$\left(\begin{array}{l} x = \cos\theta \\ dx = -\sin\theta\,d\theta \end{array} \right)$$

$$= \int_{\theta=\pi}^{\theta=0} \frac{\cos(n \arccos \cos\theta)\,\cos(m \arccos \cos\theta)}{\sqrt{1-\cos^2\theta}}(-\sin\theta)\,d\theta$$

$$= \int_{\theta=0}^{\theta=\pi} \frac{\cos n\theta\,\cos m\theta}{\sqrt{\sin^2\theta}}\,\sin\theta\,d\theta$$

$$\left(\theta \in (0,\pi) \Rightarrow \sin\theta \geq 0 \right)$$

$$= \int_{\theta=0}^{\theta=\pi} \cos n\theta\,\cos m\theta\,d\theta$$

$$= \int_0^{\pi} \frac{1}{2}\left[\cos[(n+m)\theta] + \cos[(n-m)\theta] \right] d\theta$$

$$= 0 \quad \text{when} \quad n \neq m, \ n, m \geq 1, \ n, m \in \mathbb{N}$$

since for $k \in \mathbb{Z} \setminus \{0\}$

$$\int_0^{\pi} \cos k\theta \, d\theta = \frac{1}{k} \sin k\theta \Big|_0^{\pi}$$

$$= 0.$$

page 170 #10 (4.5.10)

(a) $\tilde{T}_3(x) = \frac{1}{2^{3-1}} \cos(3 \arccos x)$,

so we need $3 \arccos x = 0$ or π or 2π or 3π,

i.e. $\arccos x = 0$ or $\frac{\pi}{3}$ or $\frac{2\pi}{3}$ or π

i.e. $x = \cos(0)$ or $\cos\left(\frac{\pi}{3}\right)$ or $\cos\left(\frac{2\pi}{3}\right)$ or $\cos\pi$,

i.e. $x_0 = 1$, $x_1 = \frac{1}{2}$, $x_2 = \frac{-1}{2}$, $x_3 = -1$.

(b) Suppose that q is a monic degree 3 polynomial such that

$$\max_{x \in [-1,1]} |q(x)| < \frac{1}{4}.$$

We know that $\tilde{T}_3(x) = \frac{1}{2^{3-1}} \cos(3 \arccos x)$ has $\max_{x \in [-1,1]} |\tilde{T}_3(x)| = \frac{1}{4}$ and $\tilde{T}_3$ is also a monic cubic polynomial.

Now we will see how no such q can exist:

c), d) Were such a q to exist, then R defined by $R(x) = \tilde{T}_3(x) - q(x)$ ~~would~~ would be a polynomial, and further, since both $\tilde{T}_3$ and q are monic cubic, R would have degree 2 or less.

By $\max_{x \in [-1,1]} |q(x)| < \frac{1}{4}$, we have

$$|q(x_0)|,\ |q(x_1)|,\ |q(x_2)|, \text{ and } |q(x_3)| < \frac{1}{4}$$

whereas ~~for~~ $\tilde{T}_3(x_0) = \frac{1}{4}$, $\tilde{T}_3(x_1) = \frac{-1}{4}$,

$\tilde{T}_3(x_2) = \frac{1}{4}$, and $\tilde{T}_3(x_3) = \frac{-1}{4}$.

Thus $R(x_0) = \tilde{T}_3(x_0) - q(x_0) > \tilde{T}_3(x_0) - |q(x_0)| > 0$,

$R(x_1) = \tilde{T}_3(x_1) - q(x_1) < \tilde{T}_3(x_1) + |q(x_1)| < 0$, and

similarly $R(x_2) > 0$ and $R(x_3) < 0$.

Since R is a polynomial it is continuous,
and $R(x_3) < 0$ and $R(x_2) > 0$
so by the intermediate value theorem
R must have a root between x_3 and x_2.
Similarly $R(x_2) > 0$ and $R(x_1) < 0$ so
R has a root between x_2 and x_1.
Finally $R(x_1) < 0$ and $R(x_0) > 0$
so R has a root between x_1 and x_0.
The intervals (x_3, x_2), (x_2, x_1), and (x_1, x_0)
are disjoint so these three roots of R
are distinct.

Yet R is a polynomial of degree ≤ 2,
so the only way it can have 3 distinct roots
is if it is the zero polynomial $R = 0$.
But then $\tilde{T}_3(x) = q(x)$ by $R = \tilde{T}_3 - q$.
This contradicts $\max_{x \in [-1,1]} |q(x)| < \frac{1}{4}$ since
$\max_{x \in [-1,1]} |\tilde{T}_3(x)| = \frac{1}{4}$. Thus the existence of
such a monic cubic polynomial q with $\max_{x \in [-1,1]} |q(x)| < \frac{1}{4}$
is contradictory: no such q exists.

page 177 #2 (4.6.2)

The interpolation nodes for the linear near-minimax approximation are the roots of $T_2(x) = \cos(2\arccos x)$, i.e.

$$2\arccos x = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}, \text{ i.e.}$$

$$\arccos x = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}, \text{ i.e.}$$

$$x = \cos\frac{\pi}{4} \text{ or } \cos\frac{3\pi}{4}, \text{ i.e.}$$

$$x = \frac{\sqrt{2}}{2} \text{ or } -\frac{\sqrt{2}}{2}.$$

Thus $x_0 = \frac{\sqrt{2}}{2}$ and $x_1 = \frac{-\sqrt{2}}{2}$.

The linear polynomial that agrees with $f(x) = e^x$ at these x_0, x_1 is

$$p(x) = e^{\frac{\sqrt{2}}{2}} + \frac{e^{\frac{\sqrt{2}}{2}} - e^{-\frac{\sqrt{2}}{2}}}{\sqrt{2}}\left(x - \frac{\sqrt{2}}{2}\right),$$

so p is the linear near-minimax approximation for $f(x) = e^x$ on $[-1,1]$.

$$p(x) = 1.260591837 + 1.08544164x$$

page 177, #4 (4.6.4)

Suppose that $g : [a,b] \to \mathbb{R}$ is a function.

Then we can define a new function $f : [-1,1] \to \mathbb{R}$ by

$$f(x) = g\left(\frac{(b+a) + x(b-a)}{2} \right).$$

Suppose $p : [-1,1] \to \mathbb{R}$ is a polynomial that uniformly approximates f to within ε,

i.e. $\max\limits_{x \in [-1,1]} |f(x) - p(x)| < \varepsilon$.

Define $\varphi : [a,b] \to [-1,1]$ by

$$\varphi(t) = \frac{2t - (a+b)}{b-a}$$ and define

~~[illegible]~~ $\theta : [-1,1] \to [a,b]$ by

$$\theta(x) = \frac{b+a + x(b-a)}{2}.$$

Notice φ and θ are inverses of each other

and $f = g \circ \theta$ so $f \circ \varphi = g$.

Thus $\forall t \in [-1,1]$ $\quad |g(t) - (p \circ \varphi)(t)| =$

$$|g(t) - p(\varphi(t))|$$

$$= |(f \circ \varphi)(t) - p(\varphi(t))|$$

$$= |f(\varphi(t)) - p(\varphi(t))|$$

$$\leq \max_{x \in [-1,1]} |f(x) - p(x)|$$

so $\displaystyle \max_{t \in [a,b]} |g(t) - (p \circ \varphi)(t)|$

$$\leq \max_{x \in [-1,1]} |f(x) - p(x)|.$$

Similarly $\forall x \in [-1,1]$ $\quad |f(x) - p(x)| =$

$$|g(\theta(x)) - (p \circ \varphi)(\theta(x))|$$

$$\leq \max_{t \in [a,b]} |g(t) - (p \circ \varphi)(t)| \quad \text{so}$$

$$\max_{x \in [-1,1]} |f(x) - p(x)| \leq \max_{t \in [a,b]} |g(t) - (p \circ \varphi)(t)|,$$

so actually $\displaystyle \max_{x \in [-1,1]} |f(x) - p(x)| = \max_{t \in [a,b]} |g(t) - (p \circ \varphi)(t)|.$

Thus, if p is a polynomial of degree n that uniformly approximates f on $[-1,1]$ to within ε, then $p \circ \varphi$ is also a polynomial of degree n (since φ is a degree 1 polynomial)) which uniformly approximates g to within ε.

If $g(t) = e^t$ on $[0,1]$,

then $f(x) = e^{\frac{x+1}{2}}$ on $[-1,1]$.

The near-minimax cubic approximation for f is that cubic polynomial p that agrees with f at

$x_0 = \cos\frac{\pi}{8}$, $x_1 = \cos\frac{3\pi}{8}$, $x_2 = \cos\frac{5\pi}{8}$, and $x_3 = \cos\frac{7\pi}{8}$,

the zeroes of T_4.

In particular

$$p(x) = 1.648180088 + 0.8243066456x + 0.210415131x^2 + 0.034779957 2x^3.$$

Thus $(p \circ \varphi)(t) = p(2t-1) =$

$$0.9995086162 + 1.01563251t + 0.4243010376t^2 + 0.2782396576t^3$$

is the near-minimax cubic approximation of $t \mapsto e^t$ on $[0,1]$.

Another equivalent way of finding the cubic ~~mo~~ near minimax approximating polynomial on $[0,1]$ for $t \mapsto e^t$ is to find the cubic that agrees with $t \mapsto e^t$ at the points

$$t_0 = \theta(x_0) = \frac{1+x_0}{2} = 0.961939766\overline{3}$$

$$t_1 = \theta(x_1) = \frac{1+x_1}{2} = 0.6913417162$$

$$t_2 = \theta(x_2) = \frac{1+x_2}{2} = 0.308658283\overline{8}$$ and

$$t_3 = \theta(x_3) = \frac{1+x_3}{2} = 0.0380602337.$$

We find that it is

$$0.9995086158 + 1.01563251\, t + 0.4243010377\, t^2 + 0.2782396575\, t^3.$$

Notice that this is the same answer up to rounding errors that are different.

This cubic near-minimax approximation is uniformly within 6×10^{-4} of $t \mapsto e^t$ on $[0,1]$. (This was determined by graphing)

Meanwhile the ~~polynomial~~ actual -minimax approximation

$$
\begin{aligned}
m_3(x) = {} & 0.9994552 + \\
& 1.0166023x + \\
& 0.4217030x^2 + \\
& 0.2799765x^3
\end{aligned}
$$

is uniformly within 5.448×10^{-4} of $t \mapsto e^t$ on $[0,1]$ (also by graphing).

Thus the near-minimax polynomial is almost as good as the actual minimax polynomial m_3.

page 178 #10 (4.6.10)

Suppose that p is a polynomial of degree $2n$ or $2n-1$, so that we can have coefficients $a_0, a_1, a_2, \dots, a_{2n} \in \mathbb{R}$ such that

$\forall x \in \mathbb{R}$

$$p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_{2n-1} x^{2n-1} + a_{2n} x^{2n}$$

with $a_{2n} \neq 0$ if p has degree $2n$ or else $a_{2n} = 0$, $a_{2n-1} \neq 0$ if p has degree $2n-1$.

Further suppose that p is an even function, i.e. suppose $\forall x \in \mathbb{R}$ $p(-x) = p(x)$.

We claim that

$$a_1 = a_3 = a_5 = \dots = a_{2n-1} = 0.$$

Proof: Define ~~the polynomial q by~~

$$\cancel{q(x) = \frac{1}{2}\left[p(x) - p(-x)\right]}$$

$$\cancel{= \frac{1}{2}\left[a_0 + a_1 x + \dots + a_{2n-1} x^{2n-1} + a_{2n} x^{n} - (a_0 - a_1 x + a_2 x^2 - a_3 x^3 + \dots + a_{2n} x^{2n})\right]}$$

See Next page

Define the polynomial q by

$$\forall x \in \mathbb{R} \quad q(x) = \frac{1}{2}\left[p(x) - p(-x)\right]$$

$$= \frac{1}{2}\Big[a_0 + a_1 x + a_2 x^2 + \dots + a_{2n-1} x^{2n-1} + a_{2n} x^{2n}$$

$$- \left(a_0 - a_1 x + a_2 x^2 - + - + \dots - a_{2n-1} x^{2n-1} + a_{2n} x^{2n}\right)\Big]$$

$$= a_1 x + a_3 x^3 + a_5 x^5 + \dots + a_{2n-1} x^{2n-1}.$$

Notice that $\forall x \in \mathbb{R}$ $q(x) = 0$ since

$$q(x) = \frac{1}{2}\left[p(x) - p(-x)\right] = \frac{1}{2}\left[p(x) - p(x)\right] = 0.$$

Thus every $x \in \mathbb{R}$ is a root of the polynomial q.

It is known that the only polynomial with more ~~tha~~ than finitely many roots is the zero polynomial, i.e. the polynomial with all of its coefficients equal to zero.

Thus q's coefficients are all equal to zero: $a_1 = 0,\ a_3 = 0,\ a_5 = 0, \dots, a_{2n-1} = 0.$

Similarly, if p is a polynomial which is odd, i.e. $p(-x) = -p(x)$,

then $q(x) \stackrel{\text{def}}{=} \frac{1}{2}[p(x) + p(-x)] =$

$= a_0 + a_2 x^2 + a_4 x^4 + \dots + a_{2n} x^{2n}$

is ~~the~~ equal to zero for every x,

so $a_0 = 0$, $a_2 = 0$, $a_4 = 0$, $\dots$, $a_{2n} = 0$.