Math 128A, Spring 2007

Homework 3 Solution

4.1.12 (a) $L_0(x)+L_1(x)+L_2(x)+L_3(x)$ is the unique polynomial of degree ≤ 3 interpolating the data $(x_0,1),(x_1,1),(x_2,1),(x_3,1)$. Since the polynomial 1 also has degree ≤ 3 and interpolates the data, we conclude

$$L_0(x)+L_1(x)+L_2(x)+L_3(x)=1.$$

(b) In general, $L_0(x)+\cdots L_n(x)$ and 1 are both degree $\leq n$ interpolants of $(x_0,1),\ldots,(x_n,1)$, and therefore must be equal.

4.1.13 For $j\leq 3$, note that $x_0^jL_0(x)+x_1^jL_1(x)+x_2^jL_2(x)+x_3^jL_3(x)$ and x^j are both polynomials of degree ≤ 3 interpolating $(x_0,x_0^j),(x_1,x_1^j),(x_2,x_2^j),(x_3,x_3^j)$. Since such interpolants are unique, the two polynomials must be equal.

For $j>3$, note that each $L_i(x)$ has degree ≤ 3, so a linear combination of them can't have the same degree as x^j.

4.1.15 Note: this is the complicated solution.

In analogy to the Lagrange basis $\{L_i(x)\}$, we want to find a basis $\{H_i(x)\}$ for the (vector space of) polynomials of degree ≤ 3 so that the coordinates of $P(x)$ with respect to $\{H_i(x)\}$ are (y_0,y_0',y_1,y_1'). The analogous versions of the requirements

$$L_i(x_j)=\delta_{ij}$$

are

$$\begin{aligned} H_{2i}(x_j)&=\delta_{ij} & H_{2i}'(x_j)&=0\\ H_{2i+1}(x_j)&=0 & H_{2i+1}'(x_j)&=\delta_{ij}\end{aligned}$$

These insure, for example, that $H_0(x)$ controls the value of $P(x)$ at 0, but doesn't affect the value of $P(x)$ at 1 or the derivative of $P(x)$ at 0 or 1.

The requirements are met by the polynomials

$$H_{2i}(x)=(1-2L_i'(x_i)(x-x_i))L_i^2(x) \text{ and } H_{2i+1}(x)=(x-x_i)L_i^2(x)$$

where $L_0(x)$ and $L_1(x)$ are the usual Lagrange basis elements. I.e.

$$\begin{aligned} H_0(x)&=\left(1-\frac{2}{x_0-x_1}(x-x_0)\right)\left(\frac{x-x_1}{x_0-x_1}\right)^2\\ H_1(x)&=(x-x_0)\left(\frac{x-x_1}{x_0-x_1}\right)^2\\ H_2(x)&=\left(1-\frac{2}{x_1-x_0}(x-x_1)\right)\left(\frac{x-x_0}{x_1-x_0}\right)^2\\ H_3(x)&=(x-x_1)\left(\frac{x-x_0}{x_1-x_0}\right)^2\end{aligned}$$

and

$$P(x) = y_0 H_0(x) + y_0' H_1(x) + y_1 H_2(x) + y_1' H_3(x)$$

4.1.16 The divided differences are easy to compute

i	x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$
0	0	-1	0	2
1	1	-1	4	
2	1	-1		

(remember that $f[x, x] = f'(x)$!) and the polynomial just as easy to read off from the top row:

$$P(x) = -1 + 0 \cdot x + 2 \cdot x(x-1)$$

The method of the previous problem is more complicated. We want polynomials $\{Q_i(x)\}$ for $i = 0, 1, 2$ which form a basis for the polynomials of degree ≤ 2 so that the coordinates of the polynomial we're looking for are $(-1, -1, 4)$. In particular, we require

$$\begin{array}{lll} Q_0(0) = 1 & Q_0(1) = 0 & Q_0'(1) = 0 \\ Q_1(0) = 0 & Q_1(1) = 1 & Q_1'(1) = 0 \\ Q_2(0) = 0 & Q_2(1) = 0 & Q_2'(1) = 1 \end{array}$$

So we find solutions:

$$Q_0(x) = (x-1)^2, Q_1(x) = 1 - (x-1)^2, \text{ and } Q_2(x) = x(x-1)$$

and then write

$$P(x) = -1 \cdot (x-1)^2 - 1 \cdot (1 - (x-1)^2) + 4 \cdot x(x-1)$$

4.1.28 Because we have 6 data points and the original polynomial has degree ≤ 5, we know that the interpolating polynomial is the original polynomial.

The divided difference algorithm:

i	x_i	$f[x_i]$	$\cdots$				
0	-2	-5	6	-3	1	0	0
1	-1	1	0	0	1	0	
2	0	1	0	3	1		
3	1	1	6	6			
4	2	7	18				
5	3	25					

yields

$$p(x) = -5 + 6(x+2) - 3(x+2)(x+1) + 1(x+2)(x+1)x$$

so $p(x)$ has degree 3.

4.1.31 The key observation is: for interpolation on $m+1$ nodes, $f[x_0, \ldots, x_m]$ is the coefficient of x^m in the interpolating polynomial.

For $m \geq n$, we know the interpolating polynomial $p(x)$ is just the original polynomial x^n, i.e. $p(x) = x^n$. If $m = n$, then the coefficient of x^m in x^n is 1 and thus $f[x_0, \ldots, x_m] = 1$. If $m > n$, then the coefficient of x^m in x^n is 0 so $f[x_0, \ldots, x_m] = 0$.

4.2.3 We just need to apply the error formula for Lagrange interpolation and then simplify.

(a) For $x \in [x_i, x_{i+1}]$ we know there is some $c \in [x_i, x_{i+1}]$ so that

$$f(x) - P_1(x) = \frac{(x - x_i)(x - x_{i+1})}{2} f''(x)$$

In our particular case,

$$|f''(c)| = |2ce^{-c^2}| \leq 2$$

and

$$\left| \frac{(x - x_i)(x - x_{i+1})}{2} \right| \leq \max_{x \in [0, 0.01]} \frac{x(0.01 - x)}{2} = \frac{0.05^2}{2}$$

so

$$|f(x) - P_1(x)| \leq 0.05^2 \approx 1.25 \times 10^{-3}$$

(b) As in (a), we know

$$f(x) - P_2(x) = \frac{(x - x_i)(x - x_{i+1})(x - x_{i+2})}{6} f'''(c)$$

and then for our specific case

$$|f'''(c)| = |(2 - 4c)e^{-c^2}| \leq 2$$

and

$$\left| \frac{(x - x_i)(x - x_{i+1})(x - x_{i+2})}{6} \right| \leq \max_{x \in [0, 0.02]} \frac{|x(x - 0.01)(x - 0.02)|}{6} = \frac{0.01^3}{9\sqrt{3}}$$

so

$$|f(x) - P_2(x)| \leq \frac{0.01^3}{9\sqrt{3}} 2 \approx 1.28 \times 10^{-7}$$

4.2.13 Induct on n. The base case already done, so consider

$$|(x - x_0) \cdots (x - x_{n+1})|$$

If $x \in [x_0, x_n]$, then by induction hypothesis

$$|(x - x_0) \cdots (x - x_n)| \leq n! h^{n+1}$$

and, since the x_is have even spacing h,

$$|x - x_{n+1}| \leq |x_0 - x_{n+1}| = (n+1)h$$

Putting these together yields

$$|(x - x_0) \cdots (x - x_n)| \leq (n+1)! h^{n+2}$$

If instead $x \in [x_1, x_{n+1}]$, a symmetric argument yields the same conclusion.

4.2.14 We compute

$$\begin{aligned} |e^x - P_n(x)| &\leq \left| \frac{(x - x_0) \cdots (x - x_n)}{n!} e^c \right| \\ &\leq \frac{n!(1/n)^{n+1}}{n!} e^c \leq (1/n)^{n+1} e^c \leq (1/n)^{n+1} e \to 0 \end{aligned}$$