Solutions to HW 2

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page 96 # 1 (3.3.1)

We want to find a zero of $f(x) \stackrel{def}{=} x^3 - x^2 - x - 1$ using the secant method.

We have to choose two guesses for the root. $x_0 = 1$ and $x_1 = 2$ were popular guesses. We iterate

$$x_{n+1} = x_n - f(x_n) \cdot \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

until we find consecutive x's x_{k-1} and x_k that differ by less than 10^{-6}:

$x_0 = 1$

$x_1 = 2$

$x_2 = 1.666...$

$x_3 = 1.816..$

$x_4 = 1.842...$

$x_5 = 1.8392156$

$$\left.\begin{array}{l} x_6 = 1.8392865 3794 \\ x_7 = 1.8392867 5523 \end{array}\right\} |x_7 - x_6| < 10^{-6}$$

So our answer is $x_7 = 1.8392867...$

page 106 # 1ab

(a) Starting with $x_0 = 1$ we iterate

$x_{n+1} = 1 + 0.3 \sin(x_n)$ to get

$x_0 = 1$

$x_1 = 1.252\ldots$

$x_2 = 1.2849\ldots$

$x_3 = 1.2878\ldots$

$x_4 = 1.28806\ldots$

$x_5 = 1.288089\ldots$

$x_6 = 1.28809 1157\ldots$. This and all

other guesses for x_0 all converge

to the same fixed point $1.28809 1313\ldots$

(b) $f(x) = 1 + 0.3 \sin(x)$ has

$0.7 \le x \le 1.3 \Rightarrow 0.7 \le f(x) \le 1.3$

since $|\sin x| \le 1$. f and f' are continuous

and $f'(x) = 0.3 \cos(x)$ has

$0.7 \le x \le 1.3 \Rightarrow |f'(x)| \le 0.3$

since $|\cos x| \le 1$. Thus the interval

$[a, b] = [0.7, 1.3]$ satisfies the hypotheses of

Theorem 3.4.2.

Problem 8, page 107 (3.4.8)

ⓐ Let $g(x) = \dfrac{15x^2 - 24x + 3}{4x}$.

Then $g'(x) = \dfrac{(30x-24)(4x) - (15x^2-24x+3)(4)}{(4x)^2}$

so $g'(\alpha) = g'(1) = \dfrac{6 \cdot 4 - (-6)(4)}{16} = 3$

so $|g'(\alpha)| > 1$, so there is no way that x_0's close to α would converge under iteration to $\alpha = 1$.

ⓑ Let $g(x) = \frac{3}{4}x + x^{-3}$.

Then $g'(x) = \frac{3}{4} - 3x^{-4}$ and

$g''(x) = 12x^{-5}$ so

$g'(\alpha) = g'(\sqrt{2}) = \frac{3}{4} - \frac{3}{4} = 0$ and

$g''(\alpha) = g''(\sqrt{2}) = \frac{3}{\sqrt{2}} \neq 0$, so

for all x_0 sufficiently close to α the method converges, and it does so quadratically, i.e. the order of convergence is 2.

page 108 #14 c (3.4.14c)

Define $g(x) \stackrel{def}{=} 6.28 + \sin(x)$.

Define $\text{Aitken}(A,B,C) \stackrel{def}{=} C - \dfrac{(B-C)^2}{A-2B+C}$.

Let $x_0 = 6$

$x_1 = g(x_0) = 6.000584502$

$x_2 = g(x_1) = 6.001145771$

$x_3 = \text{Aitken}(x_0, x_1, x_2)$
$= 6.014705148$

$x_4 = g(x_3) = 6.014733649$

$x_5 = 6.014761129$

$x_6 = \text{Aitken}(x_3, x_4, x_5)$
$= 6.015500802$

$x_7 = g(x_6) = 6.015500883$

$x_8 = g(x_7) = 6.015500961$

$x_9 = \text{Aitken}(x_6, x_7, x_8) =$
$= 6.015503073$

$x_{10} = g(x_9) = 6.015503073$

Thus $x_9 \approx x_{10}$ is a fixed point of g.

page 115 # 3 (3.5.3)

Suppose that $f(x) = (x-\alpha)^m h(x)$ where $h(\alpha) \neq 0$, so that f has α as a root of multiplicity m. Then the newton iteration function

$$g(x) \stackrel{def}{=} x - \frac{f(x)}{f'(x)} =$$

$$= x - \frac{(x-\alpha)^m h(x)}{m(x-\alpha)^{m-1} h(x) + (x-\alpha)^m h'(x)}$$

$$= x - \frac{(x-\alpha) h(x)}{m\, h(x) + (x-\alpha) h'(x)}.$$

Taking the derivative we get

$$g'(x) = 1 - \frac{\left(h(x) + (x-\alpha)h'(x)\right)\left(m h(x) + (x-\alpha) h'(x)\right)}{\left[m h(x) + (x-\alpha) h'(x)\right]^2}$$

$$+ \frac{(x-\alpha) h(x) \left[m h'(x) + (x-\alpha) h''(x) + h'(x)\right]}{\left[m h(x) + (x-\alpha) h'(x)\right]^2}$$

so that

$$g'(\alpha) = 1 - \frac{h(\alpha)\, m h(\alpha)}{[m h(\alpha)]^2} = 1 - \frac{1}{m} = \frac{m-1}{m}.$$

page 115 #3 (3.5.3) continued

since α is a fixed point of g

and $|g'(\alpha)| = \left| \frac{m-1}{m} \right| < 1$

and since g and g' are continuous in an interval containing α

(this uses that h, h', and h'' are continuous in an interval containing α and $h(\alpha) \neq 0$)

by Corollary 3.4.3 there is an interval $[a,b]$ around α for which the hypotheses, of 3.4.2 hold true.

and thus the conclusions of 3.4.2 hold true.

In particular, for any initial estimate $x_0 \in [a,b]$ the sequence given by $x_{n+1} = g(x_n)$ converges to α and moreover

$$\lim_{n\to\infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = g'(\alpha) = \frac{m-1}{m}.$$

page 115 #3 (3.5.3) continued 2

Thus $\lambda_n \overset{\text{def}}{=} \dfrac{x_n - x_{n-1}}{x_{n-1} - x_{n-2}} =$

$$= \frac{(\alpha - x_{n-1}) - (\alpha - x_n)}{(\alpha - x_{n-2}) - (\alpha - x_{n-1})}$$

$$\approx \frac{\lambda(\alpha - x_{n-2}) - \lambda(\alpha - x_{n-1})}{(\alpha - x_{n-2}) - (\alpha - x_{n-1})} = \lambda$$