Math 128A, Spring 2007

Homework 13 Solution

1. For $[-1, 1]$ and $w(x) = 1$ find the best mean-square approximation to x^3 from among polynomials of degree no bigger than 2.

   Solution: We know that the first three Legendre polynomials, 1, x, and $(3x^2 - 1)/2$, form an orthonormal basis for P_2 with respect to the inner product
   $$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$$
   So we need only compute
   $$q(x) = \langle x^3, 1\rangle 1 + \langle x^3, x\rangle x + \langle x^3, (3x^2 - 1)/2\rangle(3x^2 - 1)/2 = \frac{2}{5}x$$

2. For data $(0, -2)$, $(1, -1)$, $(2, -1)$, $(4, 3)$, $(6, 4)$ and weight function $w = (1, 2, 3, 2, 1)$, find the best mean-square linear approximation, by finding an orthonormal basis for P_1.

   Solution: The relevant inner product here is:
   $$\langle f, g\rangle = f(0)g(0) + 2f(1)g(1) + 3f(2)g(2) + 2f(4)g(4) + f(6)g(6)$$
   To find an orthonormal basis, we apply Graham-Schmidt to the basis $\{1, x\}$, i.e.
   $$w_0 = 1/||1|| = 1/\sqrt{1 + 2 + 3 + 2 + 1} = 1/3$$
   $$\hat{w}_1 = x - \langle x, w_0\rangle w_0 = x - \langle x, 1/3\rangle 1/3 = x - 22/9$$
   $$w_1 = \hat{w}_1/||\hat{w}_1|| = \hat{w}_1/(254/9) = x/254 - 11/127$$
   and then the best approximation is
   $$q(x) = \langle data, w_0\rangle w_0 + \langle data, w_1\rangle w_1$$