p. 230

$$\int_0^1 x^m \log\left(\frac{1}{x}\right) dx = \frac{1}{(m+1)^2}$$

$$\langle p_0, p_0 \rangle = \int_0^1 \log\left(\frac{1}{x}\right) dx = 1$$

$$p_0 = 1$$

$$p_1 = x - \frac{\langle x, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0 = x - \frac{1}{4}$$

since $\langle x, p_0 \rangle = \int_0^1 x \log\left(\frac{1}{x}\right) dx = \frac{1}{(1+1)^2} = \frac{1}{4}$

$$p_2 = x^2 - \frac{\langle x^2, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0 - \frac{\langle x^2, p_1 \rangle}{\langle p_1, p_1 \rangle} p_1$$

$$= x^2 - \frac{\frac{1}{9}}{1} 1 - \frac{\frac{1}{16} - \frac{1}{4} \cdot \frac{1}{9}}{\frac{1}{9} - 2 \cdot \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{16}} \left(x - \frac{1}{4}\right)$$

since $\langle x^2, p_1 \rangle = \int_0^1 x^2 \left(x - \frac{1}{4}\right) \log\left(\frac{1}{x}\right) dx = \frac{1}{16} - \frac{1}{4} \frac{1}{9}$ and

$\langle p_1, p_1 \rangle = \int_0^1 \left(x^2 - 2 \cdot \frac{1}{4} \cdot x + \frac{1}{16}\right) \log\left(\frac{1}{x}\right) dx = \frac{1}{9} - 2 \cdot \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{16}$

$$= x^2 - \frac{1}{9} - \frac{5}{7}\left(x - \frac{1}{4}\right) = x^2 - \frac{5}{7} x + \frac{17}{252}$$

Since

$$p_2 = x^2 - \frac{5}{7}x + \frac{17}{252}$$

its roots are

$$\frac{\frac{5}{7} \pm \sqrt{\frac{25}{49} - 4 \cdot \frac{17}{252}}}{2} = \frac{\frac{5}{7} \pm \sqrt{\frac{106}{441}}}{2}$$

$$= \frac{\frac{5}{7} \pm \frac{1}{21}\sqrt{106}}{2} = \frac{15 \pm \sqrt{106}}{42}$$

Part b So the quadrature of degree of ~~deg~~ precision 3

$= 2n+1$ for $n=1$ will be

$$\int_0^1 f(x)\log\left(\frac{1}{x}\right)dx \approx \sum_{k=0}^{1} A_k f(c_k)$$

where $c_0 = \frac{15-\sqrt{106}}{42}$ and $c_1 = \frac{15+\sqrt{106}}{42}$ are

the roots of $p_{n+1} = p_2$.

But who are the A's?

Let $L_0(x) \stackrel{\text{def}}{=} \dfrac{x - c_1}{c_0 - c_1}$, and let

$$L_1(x) \stackrel{\text{def}}{=} \frac{x - c_0}{c_1 - c_0}.$$

Then $A_0 = \int_0^1 L_0(x) \log\left(\frac{1}{x}\right) dx$

$$= \frac{\frac{1}{4} - c_1}{c_0 - c_1} = 0.718539319\ldots = \frac{21}{\sqrt{106}}\left(c_1 - \frac{1}{4}\right)$$

and $A_1 = \int_0^1 L_1(x) \log\left(\frac{1}{x}\right) dx =$

$$= \frac{\frac{1}{4} - c_0}{c_1 - c_0} = 0.281460681\ldots = 1 - A_0$$

So the answer is

$\int_0^1 f(x) \log\left(\frac{1}{x}\right) dx \approx A_0 f(c_0) + A_1 f(c_1)$ where

$A_0 = 0.718539319.$
$A_1 = 0.281460681\ldots$
$c_0 = 0.1120088062\ldots$
$c_1 = 0.6022769081\ldots$.

p. 230 part b Demonstration

~~Th~~ Try, for example

$$f(x) = 5x^3 - 2x^2 + x + 3$$

$$\int_0^1 (5x^3 - 2x^2 + x + 3) \log\left(\frac{1}{x}\right) dx = \frac{481}{144}$$

$$= 3.340277778\ldots$$

Where as

$$\sum_{k=0}^{1} A_k f(c_k) = A_0 f(c_0) + A_1 f(c_1)$$

$$= \overset{0.718539319}{\cancel{0.281460681}} \cdot (5c_0^3 - 2c_0^2 + c_0 + 3)$$

$$+ \underset{0.281460681}{\cancel{0.718539319}} \cdot (5c_1^3 - 2c_1^2 + c_1 + 3)$$

$$= 3.340277778$$ which is exactly right!!

Define $\langle f, g \rangle \stackrel{def}{=} \int_{-1}^{1} f(x) g(x) (1-x^2)^{-\frac{1}{2}} dx$

$$= \int_{-1}^{1} \frac{f(x) g(x)}{\sqrt{1-x^2}} dx ,$$

Lets Gram-Schmidt $1, x, x^2, x^3 \ldots$
with respect to $\langle \cdot, \cdot \rangle$.

$$p_0 = 1$$

$$p_1 = x - \frac{\langle x, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0$$

$$= x - 0 = x$$

since $\langle x, p_0 \rangle = \int_{-1}^{1} \frac{x \cdot 1}{\sqrt{1-x^2}} dx = 0$

by oddness of the integrand.

$$p_2 = x^2 - \frac{\langle x^2, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0 - \frac{\overbrace{\langle x^2, p_1 \rangle}^{=0}}{\langle p_1, p_1 \rangle} p_1$$

$$=$$

By $x=\cos\theta$ $dx=-\sin\theta\, d\theta$ so

$$\int_{-1}^{1}\frac{x^2}{\sqrt{1-x^2}}\,dx = \int_{\theta=\pi}^{\theta=0}\frac{\cos^2\theta}{\sqrt{\sin^2\theta}}(-\sin\theta)\,d\theta$$

$$=\int_{\theta=0}^{\theta=\pi}\cos^2\theta\,d\theta = \int_0^{\pi}\frac{1-\cos 2\theta}{2}\,d\theta$$

$$=\frac{\pi}{2} = \langle x^2, p_0\rangle \overset{\text{and also}}{=} \langle p_1, p_1\rangle$$

and thus

$$p_2 = x^2 - \frac{\frac{\pi}{2}}{\pi}\,1 = x^2 - \frac{1}{2}$$

$$p_3 = x^3 - \frac{\overset{=0}{\langle x^3, p_0\rangle}}{\langle p_0, p_0\rangle}p_0 - \frac{\langle x^3, p_1\rangle}{\langle p_1, p_1\rangle}p_1 - \frac{\overset{=0}{\langle x^3, p_2\rangle}}{\langle p_2, p_2\rangle}p_2$$

$$= x^3 - \frac{\int_{-1}^{1}\frac{x^4}{\sqrt{1-x^2}}dx}{\frac{\pi}{2}}\,x = x^3 - \frac{\frac{3\pi}{8}}{\frac{\pi}{2}}\,x$$

$$= x^3 - \frac{3}{4}x$$ because

$$\int_{-1}^{1} \frac{x^4}{\sqrt{1-x^2}}\,dx = \int_{0}^{\pi} \frac{\cos^4\theta}{\sqrt{1-\cos^2\theta}} \sin\theta\, d\theta$$

$$= \int_{0}^{\pi} \cos^4\theta\, d\theta = \int_{0}^{\pi} \left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta$$

$$= \int_{0}^{\pi} \frac{1}{4} - \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos^2 2\theta\, d\theta$$

$$= \frac{\pi}{4} - 0 + \frac{1}{4}\int_{0}^{\pi} \frac{1-\cos 4\theta}{2}\, d\theta$$

$$= \frac{\pi}{4} + \frac{1}{4}\cdot\frac{\pi}{2} = \frac{3\pi}{8}$$

$$P_3 = x^3 - \frac{3}{4}x$$

$$= x\left(x^2 - \frac{3}{4}\right) = x\left(x - \frac{\sqrt{3}}{2}\right)\left(x + \frac{\sqrt{3}}{2}\right)$$

so

$$c_0 = -\frac{\sqrt{3}}{2}, \quad c_1 = 0, \quad c_2 = \frac{\sqrt{3}}{2}$$

and thus the quadrature is

$$\int_{-1}^{1} f(x)\,(1-x^2)^{-\frac{1}{2}}\,dx \approx \sum_{k=0}^{2} A_k f(c_k)$$

where $A_k = \int_{-1}^{1} L_k(x)\,(1-x^2)^{-1/2}\,dx$ where

$$L_0(x) = \frac{x\left(x - \frac{\sqrt{3}}{2}\right)}{-\frac{\sqrt{3}}{2}\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)} = \frac{2}{3}x\left(x - \frac{\sqrt{3}}{2}\right)$$

$$L_1(x) = \frac{\left(x + \frac{\sqrt{3}}{2}\right)\left(x - \frac{\sqrt{3}}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right)} = \frac{4}{3}\left(\frac{3}{4} - x^2\right)$$

$$= 1 - \frac{4}{3}x^2$$

$$L_2(x) = \frac{\left(x + \frac{\sqrt{3}}{2}\right)x}{\sqrt{3}\cdot\frac{\sqrt{3}}{2}} = \frac{2}{3}x\left(x + \frac{\sqrt{3}}{2}\right)$$

$$A_0 = \int_{-1}^{1} \left(\frac{2}{3}x^2 - \frac{\sqrt{3}}{3}x\right)\frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \frac{2}{3}\cdot\frac{\pi}{2} \qquad = \frac{\pi}{3} =$$

$$A_1 = \int_{-1}^{1} \left(1 - \frac{4}{3}x^2\right)\frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \pi - \frac{4}{3}\cdot\frac{\pi}{2} = \frac{\pi}{3}$$

$$A_2 = \int_{-1}^{1} \left(\frac{2}{3}x^2 + \frac{\sqrt{3}}{3}x\right)\frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \frac{2}{3}\cdot\frac{\pi}{2} = \frac{\pi}{3}$$

So the quadrature is

$$\int_{-1}^{1} f(x)\,(1-x^2)^{-\frac{1}{2}}\,dx \approx \frac{\pi}{3}\left[f\left(\frac{-\sqrt{3}}{2}\right) + f(0) + f\left(\frac{\sqrt{3}}{2}\right)\right].$$